1st PICASSO conference
France
Portugal
Spain

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- **Carlos Parés**

Website
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Bem-vindo ¡Bienvenido! Bienvenus! Welcome!

Welcome to the 1st PICASSO workshop "**hyPerbolIC models, numerical AnalysiS and Scientific cOmputation**" in Málaga, March 24-26, 2025,
a three day meeting in Málaga, Spain, to celebrate Spain - Portugal - France privileged relationship for researchers involved in modeling, numerical analysis and scientific computation.

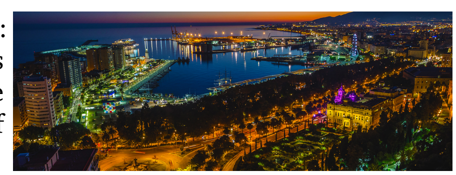
It coincides with the newly created **International Research Project (IRP)** also called PICASSO, a French CNRS initiative, which can be seen as a virtual Laboratory between us.

The common topics are wide and not restricted to

- Nonlinear Partial Differential Equations (PDEs);
- High performing numerical schemes;
- Numerical analysis of models;
- High-performance computing;
- Real-life applications;
- Modeling and simulation of geophysical flows.

The Mediteranneen sea, the Andalusian history and culture associated to a nice weather in spring lead to a perfect incubator for exceptional interactions and great science!

Three day workshop in beautiful Málaga



Your workshop is organized as follows: Presentations from different groups and researchers during the day are interspersed with large moments of scientific interactions.

Only ∼60 people are expected to this workshop to encourage scientific interactions. You should leave the workshop with new collaborators, new projects, more advanced ones, ongoing articles and a scientific momentum to reach next year PICASSO workshop!

UNIVERSIDADE D COIMBRA · UNIVERSIDAD DE MÁLAGA · université de BORDEAUX · UNIVERSITÉ DE NANTES · UVSQ université PARIS-SACLAY · cnrs · EDANYA

PICASSO International Research Project

ORIGINS. In 2024 some researchers from France, Spain in Andalusia and Portugal realized that our scientific relationship was really strong in the "broad" themes of modeling, numerical analysis and scientific computation.
For decades a large amount of researchers have intensively collaborated, exchanging, co-mentoring students, visiting each others, organizing joint events, building projects, and, most of all, writing breaking-through articles. However as a 'community' this relationship was never truly formalized into any administrative structure.The French CNRS gave us the opportunity of highlighting this relationship with the creation of a so-called International Research Project.

WHAT IS AN IRP? An IRP aims at *structuring an international scientific community around a shared theme. It promotes the organization of international workshops and seminars or thematic schools organized by network partners, in France and abroad. Lasting 5 years, it brings together researchers from French laboratories, and several partner laboratories abroad. An IRP receives specific funding from CNRS for international mobility between the laboratories, and for the organization of international workshops/seminars, working meetings and thematic schools organized by the partners.*

A FLASH OF INSPIRATION! The idea of devising an IRP called PICASSO emerged in our minds during the ice-breaking cocktail before the banquet of HYP conference which took place in Málaga in June 2022. Málaga is the birth place of Pablo Picasso (1881-1973), the ultra-famous painter, Francophile and a path-finder in art. A dedicated museum spanning the life of P. Picasso opened in 2003 in Málaga. Therefore, rather naturally, during this pre-drink session in June 2022 we came with the idea to name this IRP "PICASSO". Later we found that it may stand for "**hyPerbolIC models, numerical AnalysiS and Scientific cOmputation**".

PICASSO IRP, BORN JANUARY THE 1ST, 2025.
The French CNRS has officially approved PICASSO in 2025 and for 5 years. Althoug all researchers from our three countries involved in those themes are naturally part of PICASSO, we had to structure the project with three "hubs" laboratories, one in Málaga , one in Coimbra and one in Bordeaux. The support from many other mathematical partners has proven that our community is far wider! Indeed PICASSO has been possible because of the active support of researchers from Sevilla, Granada in Spain, also Minho, NOVA Lisboa in Portugal, Nantes, Versailles Saint-Quentin in France.

AXES AND THEMES. The main research theme revolves around the development and analysis of models, numerical methods and simulation codes dedicated to solve fluid flow problems pertaining from true geophysical and environmental applications.

- **Axe 1 - Hyperbolic systems of PDEs** lead by **C. Pares** (Málaga) and **C. Berthon** (Nantes)
- **Axe 2 - Modeling and simulation of geophysical flows** lead by **T. Morales de Luna** (Málaga) and **C. Chalons** (Versailles)
- **Axe 3 - High performance computing and verification** lead by **M. Castro** (Málaga) and **R. Loubère** (Bordeaux)

Obviously this list is neither exhaustive nor restrictive, it simply provides some 'boxes' where on-going collaborations are taking place and would welcome you if needed.

HOW PICASSO CAN HELP?
Soon PICASSO will have a web-site, a direction and its own funding from CNRS, intended to organise and animate our common scientific life: (short) visits and scientific workshops are two targetted actions which may be funded. Any good idea is welcomed!

HUBS.

Country	City	Laboratory	Web	Local PI
Spain	Málaga	Differential Equations, Numerical Analysis and Applications (EDANYA)	www.uma.es/edanya	Carlos Pares
Portugal	Coimbra	Centro de Matemática (CMUC)	cmuc.mat.uc.pt	Stéphane Clain
France	Bordeaux	Institut de Mathématiques de Bordeaux (IMB)	math.u-bordeaux.fr	Raphaël Loubère

The founders of PICASSO are M. Castro, T. Morales de Luna and C. Pares in Spain, C. Berthon, C. Chalons and R. Loubère in France, S. Clain, G. Machado and D. Albuquerque in Portugal. However we expect more active decision-makers in the future scientific PICASSO committee, and any goodwill is welcome!

Tentative schedule

	Monday 24th	Tuesday 25th	Wednesday 26th
9h15 - 10h00	(9h30) **PICASSO** p.7	**S. GAVRILIUK** p.26	**M. RICCHIUTO** p.53
10h00 - 10h30	A. González del Pino p.28	W. Barsukow p.13	X. Nogueira p.43
10h30 - 11h00	A. Durán p.12	R. Costa p.24	J. Cortés p.22
11h00 - 11h45	**Coffee break**	**Coffee break**	**Coffee break**
11h45 - 12h15	A. Carpio p.10	S. Busto-Ulloa p.55	D. Martinez p.39
12h15 - 12h45	M. Gómez Mármol p.34	D. Albuquerque p.8	S. Serna p.56
12h45 - 13h15	I. Cordero-Carrión p.21	J.M. Mantas Ruiz p.35	I. Gomez-Bueno p.27
13h15 - 15h00	**Lunch**	**Lunch**	**Lunch**
15h00 - 15h45	**E. FERNANDEZ-NIETO** p.25	**P. MULET** p.49	**S. CLAIN** p.15
15h45 - 16h15	V. Perrier p.50	L. Martaud p.37	S. Fernandez-Garcia p.58
16h15 - 16h45	J. Ruiz-Álvarez p.32	F. Ortegón Gallego p.45	E. Pimentel-García p.52
16h45 - 17h15	**Coffee break**	**Coffee break**	**Coffee break**
17h15 - 17h45	POSTER PITCH p.61	T. Harbreteau p.31	E. Macca p.33
17h45 - 18h15	POSTER SESSION p.61	B. Nkonga p.41	
18h15 - 20h00 20h00 - late	**Ice-Breaking** **Banquet**	**Same as the day before** **but on your own**	

The talks are scheduled to last for “at most” 30 minutes with minimum 5 minutes of questions/answers.
Abstracts are gathered in pages 7-70.
Coffee-breaks take place in the room next to the conference room.
Poster session is a sort of hybrid talk/poster. A pitch of 5 minutes is given by the author while the poster is projected in the conference room. No question is allowed but discussions are welcomed afterward.

Sponsors

We wish to thank the financial supports from the SHARK workshops, Anabase Joint Research Lab. CEA-IMB and Edanya group, the CNRS "Mathématiques" and the Universities of Bordeaux/Nantes/Versailles-Saint-Quentin in France Universidade de Málaga and the Universidade de Comibra, and, obviously your institution allowing and paying for your presence.

About 60 participants on-site

Francesco Vecil Université Clermont Auvergne, Laboratoire de Mathématiques Blaise Pascal
Martin Parisot Inria Bordeaux
Ana Carpio Universidad Complutense de Madrid
Enrique Domingo Fernández-Nieto Universidad de Sevilla
Sergey Gavrilyuk Aix-Marseille University
Macarena Gómez Mármol Universidad de Sevilla
Isabel Cordero-Carrión University of Valencia
Vincent Perrier Inrria / Pau University
Ludovic Martaud Inria Rennes
Angel Durán University of Valladolid
Mario Ricchiuto Inria Bordeaux
Saray Busto Ulloa Department of Applied Mathematics, Universidade de Santiago de Compostela
Wasilij Barsukow CNRS/Institut de Mathématiques de Bordeaux
Stéphane Clain Coimbra university
Ricardo Costa University of Minho
Pep Mulet University of Valencia
Susana Serna Universidad Autonoma de Barcelona
Duarte Albuquerque NOVA-FCT (New University of Lisbon)
José Miguel Mantas Ruiz Associate Professor. University of Granada
Juan Ruiz-Álvarez Universidad Politécnica de Cartagena
Francisco Ortegón Gallego Universidad de Cádiz
Ernesto Pimentel-García University of Málaga
Soledad Fernández García Universidad de Sevilla
Darío Martínez Universidad de Castilla-La Mancha
Xesús Nogueira Universidade da Coruña
José Antonio García Rodríguez Universidade da Coruña
Jesús Cortés Universidad de Castilla-La Mancha
Boniface Nkonga Université Nice et Inria
Angelo Iollo Université de Bordeaux & Inria
Teresa Malheiro Centro de Matemática da Universidade do Minho
Francisco Pla Martos Universidad de Castilla-La Mancha. Departamento de Matemáticas
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Maria Lopez-Fernandez University of Málaga
Samuele Rubino University of Seville
León Miguel Ávila León Universidad de Málaga
Henar Herrero Universidad de Castilla-La Mancha
Manuel J Castro Diaz Universidad de Málaga
Alejandro Ramos-Lora University of Málaga
Izarne Martínez Donato Univesitat de València
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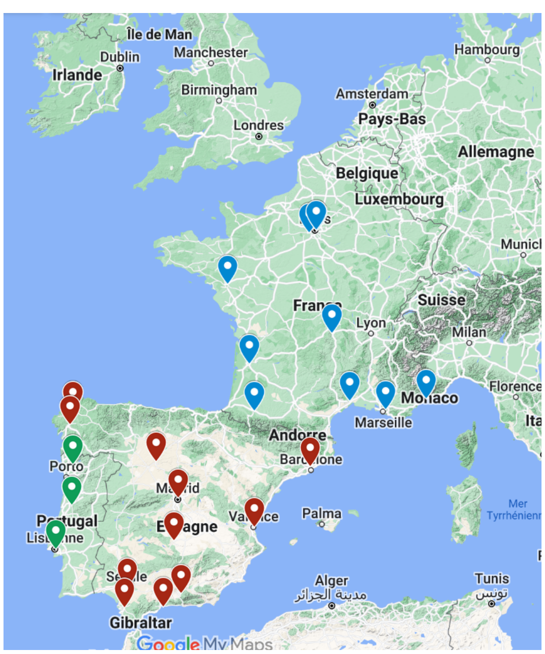
Marc de la Asunción Universidad de Málaga
Juan Francisco Rodríguez Gálvez Universidad de Málaga
Irene Gómez Bueno Universidad de Málaga
León Miguel Ávila León Universidad de Málaga
Juan Francisco Rodríguez Universidad de Málaga
Sophie Hörnschemeyer Aachen University and Sorbonne University, Germany, France



Some keywords from abstracts

POD, active flux, conservation laws, Boussinesq, Rayleigh-Bénard, CFD, sediments, path-conservative, semi-implicit, Green-Naghdi, rheology, structural scheme, very high-order, low Mach, BDF2, IMEX, discrete BC, hyperbolic, conservation of curl/div, ROM, DG, Neural Net, tsunamis, relativistic hydrodynamics, multi-state Riemann solver, (semi) implicit scheme, stiff source terms, etc.

Where are we from?

⟵ some of us

List of presentations

List of posters

INTERNATIONAL RESEARCH PROJECT PICASSO

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ABSTRACT

In this small introductionary presentation we introduce the newly created International Research Project PICASSO which is a French CNRS entity devoted to enhance the scientific relationships between France, Portugal and Spain for the large themes revolving around modeling, numerical analysis and scientific computation.

An IRP aims at structuring an international scientific community around a shared theme. It promotes the organization of international workshops and seminars or thematic schools organized by network partners, in France and abroad. Lasting 5 years, it brings together researchers from French laboratories, and several partner laboratories abroad. An IRP receives specific funding from CNRS for international mobility between the laboratories, and for the organization of international workshops/seminars, working meetings and thematic schools organized by the partners.
PICASSO Stands for "hyPerbolIC models, numerical AnalysiS and Scientific cOmputation".

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A NEW APPROACH FOR VERY HIGH-ORDER TRANSIENT SIMULATIONS WITH THE FINITE VOLUME METHOD AND OTHER ADVANCEMENTS

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ABSTRACT

In high-order finite-volume schemes, using a point-wise framework makes easier to the local polynomial reconstruction and regularizes the order decay in arbitrary unstructured grids. Although, in unsteady cases the cell mean value is required to integrate, with high accuracy, the unsteady term of the equation.
The accuracy of the point-wise based schemes is quantified for several grid types and unsteady problems. Additionally, it is shown that the stencil selection algorithm can have a significant impact the computational efficiency, both in terms of memory and time costs.
It is concluded that high-order time schemes can enhance the efficiencies of high-order spatial schemes. Furthermore, the point-wise framework is combined with immersed boundary method which reach eighth-order local accuracy, with an automatic p-adaptivity algorithm to increase the local accuracy and a sliding interface is verified with these schemes. This talk summarizes the results of the HIBforMBP project.

ACKNOWLEDGMENTS

The authors acknowledge the financial support received by FCT under the research project: High-order immersed boundary for moving body problems - HIBforMBP - with the reference PTDC/EME-EME/32315/2017. The First Author acknowledges Fundacão para a Ciência e a Tecnologia (FCT, I.P.) for its financial support via the projects UIDB/00667/2020 (UNIDEMI) and UIDP/00667/2020 (UNIDEMI).

REFERENCES

[1] A. G. R. Vasconcelos, D. M. S. Albuquerque and J. C. F. Pereira, *A Very High-Order Finite Volume Method Based on Weighted Least Squares for Elliptic Operators on Polyhedral Unstructured Grids*, Computers and Fluids, vol 181, 2019. URL: https://doi.org/10.1016/j.compfluid.2019.02.004

[2] P. M. P. Costa and D. M. S. Albuquerque *A Novel Approach for Temporal Simulations with Very High-order Finite Volume Schemes on Polyhedral Unstructured Grids*, Journal of Computational Physics, vol 435, 2022. URL: https://doi.org/10.1016/j.jcp.2022.110960

[3] Bruno F. S. Delgado, *A Simplex Immersed Boundary Method for Very High-order Finite Volume Schemes*, Master Thesis Defended at Universidade de Lisboa, 2023. URL: https://fenix.tecnico.ulisboa.pt/cursos/meaer21/dissertacao/1128253548923277

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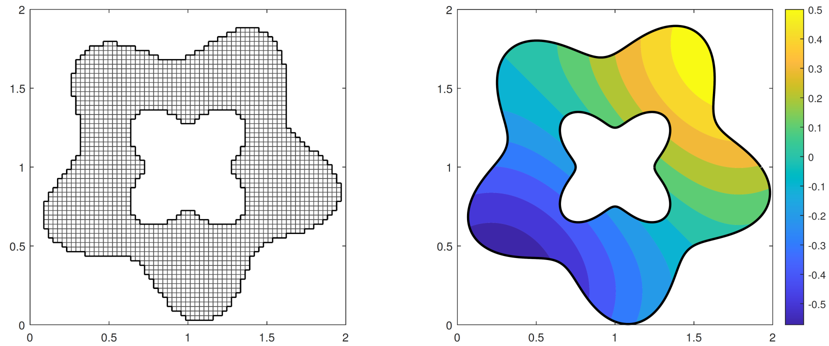

FIGURE 1: Example of a conservative cut used by the immersed boundary method (left) and distribution of the analytical solution (right) [3].

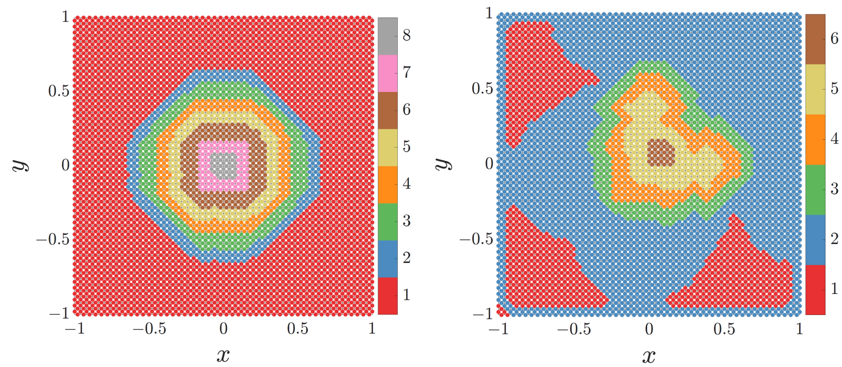

FIGURE 2: Distribution in the domain of the polynomial order used by the p-adaptivity algorithm [4].

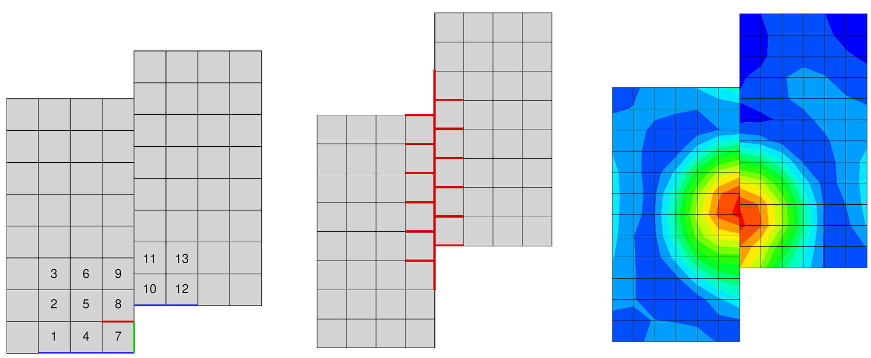

FIGURE 3: Stencil used by a forth-order scheme centered at the red face (left), example of the modified stencils in a second-order scheme (middle), and numerical error distribution at the sliding interface (right) [5].

[4] Filipe F. Tenreiro, *Order-based Adaptation Techniques for Finite Volume Schemes with Unstructured Grids*, Master Thesis Defended at Universidade de Lisboa, 2023. URL: https://fenix.tecnico.ulisboa.pt/cursos/meaer21/dissertacao/283828618790798

[5] José C. M. Ralo, *Implementation of a Sliding Interface Technique for Second and High-order Finite Volume Schemes*, Master Thesis Defended at Universidade de Lisboa, 2024. URL: https://fenix.tecnico.ulisboa.pt/cursos/meaer21/dissertacao/283828618790798

CONSERVATION LAWS SET IN MOVING DOMAINS ARISING IN CELLULAR STUDIES

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ABSTRACT

Migration and growth phenonema in cellular systems often lead to the study of systems of conservation laws set in moving domains, that is, spatial regions whose boundaries move with time. Lacking general tools to handle these problems, they are studied case by case. We will discuss some examples and the difficulties their analytical and numerical study poses. Change of variables to work in fixed domain is effective to simulate models for cell crawling, at the expense of complicating sources and coefficients. The sign of velocities allows for solution by the method of characteristics in models for biofilm eradication by antibiotic supply. In general, we face multiphase systems coupling conservation laws with convection-reaction-diffusion equations set in domains whose boundaries move as dictated by cell activity under biophysical constraints to be preserved, whose study is a challenge.

BIOFILM ERADICATION FOLLOWING CHARACTERISTIC CURVES

A one dimensional approximation of a bacterial biofilm occupying the region $0 < x < L(t)$, $t > 0$, as a biphasic mixture leads to a balance equation for the volume fraction of alive cells $\phi_l(z,t)$

$$\frac{\partial \phi_l}{\partial t} = \mu \left(\frac{C_o}{C_o + K_o} \right) \phi_l - \frac{\partial (v\phi_l)}{\partial z} - f\left(\frac{C_o}{C_o + K_o}, C_a, \phi_l \right), \quad \frac{\partial v}{\partial z} = \mu \left(\frac{C_o}{C_o + K_o} \right) \phi_l, \quad 0 < z < L(t), t > 0,$$

with initial conditions $\phi_l(z,0) = 1$, $0 \leq z \leq L(0)$, and $v(0,t) = 0$, $t > 0$, where $L(0)$ is the biofilm thickness at time $t = 0$. The biofilm thickness $L(t)$ evolves according to $L'(t) = v(L(t),t) - k_d L(t)^2$, $t > 0$, k_d being the detachment rate coefficient. The source f represents the effect of administering a concentration C_a of antibiotic in the presence of a concentration C_o of oxygen. The sign of the advection velocity makes irrelevant the motion of $L(t)$ as theoretical solutions can be constructed by the method of characteristics. Numerical simulations by means of upwind schemes allow us to quantify the antibiotic supply that would be needed to achieve finite time extinction of ϕ_l [1].

CELL CRAWLING BY CHANGE OF VARIABLE

A simple model for cell crawling leads to a one dimensional representation of the cell, which occupies a region $a(t) < x < b(t)$, $t > 0$. It is formed by two phases: a viscous fluid representing actin, with volume

*Correspondence to ana_carpio@mat.ucm.es fraction $\phi_s(x,t)$ and density ρ_s, and an inviscid fluid representing the citoplasm, with volume fraction $\phi_f(x,t) = 1-\phi_s(x,t)$, and density ρ_f. The cell movement is governed by conservation of mass and momentum

$$\rho_s \frac{\partial \phi_s}{\partial t} + \rho_s \frac{\partial}{\partial x}(\phi_s v_s) = -\rho_f \frac{\phi_s - \phi_e}{\tau}, \qquad \rho_f \frac{\partial \phi_f}{\partial t} + \rho_f \frac{\partial}{\partial x}(\phi_f v_f) = \rho_f \frac{\phi_s - \phi_e}{\tau}.$$

$$\frac{\partial}{\partial x}\left(\mu \frac{\partial v_s}{\partial x}\right) - \phi_s \frac{\partial p}{\partial x} - \frac{\partial \psi}{\partial x} + \frac{\phi_f}{k_h}(v_f - v_s) = \beta v_s, \qquad \frac{\partial}{\partial x}\left(\mu \frac{\partial v_s}{\partial x} - p - \psi\right) = \beta v_s,$$

$$v_f - v_s = -k_h \frac{\partial p}{\partial x} \qquad \psi(\phi_s) = -\psi_0((1-\phi_L)(1-\phi_R)\ln(1-\phi_s) + (1-\phi_L-\phi_R)\phi_s + (\phi_s^2/2)),$$

where v_s and v_f are the velocities of actin and citoplasm, ϕ_e the equilibrium network volume fraction, τ the time-scale for relaxation, p the pore hydrostatic pressure, ψ the contraction/swelling stress of the network, μ the actin viscosity, β a friction parameter and k_h the permeability. The change of variable

$$x = \chi(x',t) + x', \quad \phi_s(x,t) = \phi_s'(x',t), \quad v_s(x,t) = v_s'(x',t) + V'(x',t), \quad V(x,t) = V'(x',t)$$
$$0 \le x' \le 1, \quad \chi(x',t) = \int_0^t V'(x',\tau)d\tau, \quad \frac{dx'}{dx} = \frac{1}{1 + \frac{\partial}{\partial x'}\int_0^t V'(x',\tau)d\tau},$$

refers the problem to a fixed spatial domain. In spite of the complexity of the new coefficients, the resulting system can be discretized by means of quadrature rules, upwind schemes and forward-backward differentiation [4]. We obtain travelling wave-like solutions representing cell motion.

CELLULAR MODELS SET IN MOVING DOMAINS

In general, models for cell spread combine conservation laws for volume fractions of fluids and different types of cells, conservation laws for momentum and additional systems of convection-reaction-diffusion equations for relevan chemicals and signals set in domains whose boundaries move according to cell division, death and motion as wall as fluid in-flow and out-flow [2, 3]. In slab geometries, the height of the boundary $h(x_1,x_2,t)$

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x_1}\left[\int_0^h (\mathbf{v}\cdot\hat{\mathbf{x}}_1)\,dx_3\right] + \frac{\partial}{\partial x_2}\left[\int_0^h (\mathbf{v}\cdot\hat{\mathbf{x}}_2)\,dx_3\right] = \mathbf{v}\cdot\hat{\mathbf{x}}_3\big|_0,$$

where the composite velocity of the mixture $\mathbf{v} = \phi_\mathrm{f}\mathbf{v}_\mathrm{f} + \phi_\mathrm{s}\mathbf{v}_\mathrm{s}$ has components $\mathbf{v}\cdot\hat{\mathbf{x}}_i = v_{\mathrm{s},i} - k_\mathrm{h}(\phi_\mathrm{s})\frac{\partial(p-\pi)}{\partial x_i}, i = 1,2,3$, for the velocities of the different phases, the fluid and chemical pressures [2, 3]. Quasistatic approximations lead to solving the equation for h coupled to a collection of quasi-stationary problems. We present some well posedness results for such quasi-stationary approximations.

ACKNOWLEDGMENT

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REFERENCES

[1] B. Birnir, A. Carpio, G. Duro, *Driving biofilms to finite time extinction by antibiotic cocktails,* preprint, 2025.

[2] A. Carpio, G. Duro, *Well posedness of fluid-solid mixture models for biofilm spread,* Applied Mathematical Modelling 124, 64-85, 2023.

[3] A. Carpio, G. Duro, *Analysis of a two phase flow model of biofilm spread,* Nonlinear Analysis 224, 113538, 2024.

[4] R. González-Albaladejo, F. Ziebert, A. Carpio, *Two-fluid variable length model for cell crawling,* in C. Parés et al. (eds.), Hyperbolic Problems: Theory, Numerics, Applications. Volume II, SEMA SIMAI Springer Series 35, Springer Nature Switzerland AG 2024.

EXISTENCE AND STABILITY OF SOLITARY WAVE SOLUTIONS OF BOUSSINESQ-FULL DISPERSION SYSTEMS FOR INTERNAL WAVES

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ABSTRACT

This talk is concerned with the Boussinesq-Full Dispersion system. This is a three-parameter system of pde's, introduced by Bona, Lannes, and Saut in [1] as a model for the propagation of internal waves along the interface of two-fluid layers with rigid lid condition for the upper layer, and under a Boussinesq regime for the upper layer and a full dispersion regime for the lower layer. Some results on the existence and stability of solitary wave solutions will be discussed.

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REFERENCES

[1] J. L. Bona, D. Lannes, J.-C. Saut, Asymptotic models for internal waves, J. Math. Pues Appl., 89 (2008) 538-566.

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ACTIVE FLUX: A GLOBALLY CONTINUOUS, ARBITRARILY HIGH-ORDER METHOD FOR HYPERBOLIC SYSTEMS OF CONSERVATION LAWS

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ABSTRACT

The Active Flux method can be seen as an extension of the Finite Volume method: Besides the cell average it includes point values at cell interfaces. The fluxes needed for the update of the cell average thus can be immediately computed by quadrature. The point values at cell interfaces are shared among the adjacent cells (contrary to DG methods), and a piecewise continuous reconstruction is natural. The update of point values can be done in a variety of ways, but generally involves upwinding which stabilizes the method. Several extensions from the initial third-order to arbitrary order of accuracy in multiple dimensions will be shown and applications to hyperbolic systems of conservation laws, such as the Euler equations, discussed.

THE BEGINNINGS

A few years before the MUSCL scheme appears in [vL79], in [vL77] the same author discusses entirely different high-order methods for linear advection, among them "Scheme V". Besides the cell average $\bar{q}_i$, a point value $q_{i+\frac{1}{2}}$ is stored at every cell interface $x_{i+\frac{1}{2}}$. This means that every cell has access to three pieces of information (one average and two point values $q_{i\pm\frac{1}{2}}$), which allows to construct a piecewise parabolic approximation $q_{i,\text{recon}}\colon \left[-\frac{\Delta x_i}{2}, \frac{\Delta x_i}{2}\right] \to \mathbb{R}$ of the solution:

$$q_{i,\text{recon}}\left(\pm\frac{\Delta x}{2}\right) = q_{i\pm\frac{1}{2}} \qquad\qquad \frac{1}{\Delta x_i}\int_{-\frac{\Delta x_i}{2}}^{\frac{\Delta x_i}{2}} q_{i,\text{recon}}(x)\,\mathrm{d}x = \bar{q}_i \tag{1}$$

where $\Delta x_i := x_{i+\frac{1}{2}} - x_{i-\frac{1}{2}}$. It naturally is continuous and eventually gives rise to a third-order method.

Integrating the conservation law $\partial_t q + \partial_x f(q) = 0$ over the cell $[x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}]$ immediately yields the update equation for the cell average by virtue of Gauss' law:

$$\frac{\mathrm{d}}{\mathrm{d}t}\bar{q}_i + \frac{f(q_{i+\frac{1}{2}}) - f(q_{i-\frac{1}{2}})}{\Delta x_i} = 0 \tag{2}$$

This equation is *exact*, i.e. its accuracy is entirely determined by the accuracy of the point value updates.

Scheme V in [vL77] is designed specifically for linear advection $f(q) = cq$, $c > 0$, and it is natural to evolve the point values using characteristics:

$$q^{n+1}_{i+\frac{1}{2}} = q^n_{i,\text{recon}}\left(\frac{\Delta x}{2} - c\Delta t\right) \tag{3}$$

One observes the natural inclusion of upwinding. To achieve a fully discrete method it was suggested to replace (2) with a quadrature in time. The resulting method is stable with a maximum CFL $:= c\Delta t/\min \Delta x_i$ of 1.

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NONLINEAR PROBLEMS

In [ER11, ER13] this method has been applied to nonlinear and multi-dimensional problems for the first time, and named *Active Flux* (AF). Characteristic-type approximate point value updates were developed therein and also in e.g. [Fan17, BHKR19, Bar21]. Once obtained they yield an extremely efficient method with a large domain of stability. In [BB23], for instance, well-balanced AF methods for the shallow water equations in 1-d were developed. However, for multi-dimensional nonlinear problems sufficiently high-order accurate point value updates are difficult to obtain, with some recent work in this direction being [CHLM24].

SEMI-DISCRETE ACTIVE FLUX

In [Abg22, AB23] it has been suggested to replace the update of point values by a semi-discretization

$$\frac{\mathrm{d}}{\mathrm{d}t} q_{i+\frac{1}{2}} = -J(q_{i+\frac{1}{2}})^{+} R^{+}(q_{i-\frac{1}{2}}, \bar{q}_i, q_{i+\frac{1}{2}}) - J(q_{i+\frac{1}{2}})^{-} R^{-}(q_{i+\frac{1}{2}}, \bar{q}_{i+1}, q_{i+\frac{3}{2}}) \tag{4}$$

where $R^{\pm}$ is a finite-difference-like approximation of $\partial_x q$, $J = \nabla_q f$ is the Jacobian and the superscripts denote its positive/negative parts (on the eigenvalues). Together with (2) this system can be integrated in time using standard Runge-Kutta methods, explicitly or implicitly ([BB24]).

REFERENCES

[AB23] Remi Abgrall and Wasilij Barsukow. Extensions of Active Flux to arbitrary order of accuracy. *ESAIM: Mathematical Modelling and Numerical Analysis*, 57(2):991–1027, 2023.

[Abg22] Rémi Abgrall. A combination of Residual Distribution and the Active Flux formulations or a new class of schemes that can combine several writings of the same hyperbolic problem: application to the 1d Euler equations. *Communications on Applied Mathematics and Computation*, pages 1–33, 2022.

[Bar21] Wasilij Barsukow. The active flux scheme for nonlinear problems. *Journal of Scientific Computing*, 86(1):1–34, 2021.

[BB23] Wasilij Barsukow and Jonas P Berberich. A well-balanced Active Flux method for the shallow water equations with wetting and drying. *Communications on Applied Mathematics and Computation*, pages 1–46, 2023.

[BB24] Wasilij Barsukow and Raul Borsche. Implicit active flux methods for linear advection. *Journal of Scientific Computing*, 98(3):52, 2024.

[BHKR19] Wasilij Barsukow, Jonathan Hohm, Christian Klingenberg, and Philip L Roe. The active flux scheme on Cartesian grids and its low Mach number limit. *Journal of Scientific Computing*, 81(1):594–622, 2019.

[CHLM24] Erik Chudzik, Christiane Helzel, and Mária Lukáčová-Medvid'ová. Active Flux methods for hyperbolic systems using the method of bicharacteristics. *Journal of Scientific Computing*, 99(1):16, 2024.

[ER11] Timothy A Eymann and Philip L Roe. Active flux schemes for systems. In *20th AIAA computational fluid dynamics conference*, 2011.

[ER13] Timothy A Eymann and Philip L Roe. Multidimensional active flux schemes. In *21st AIAA computational fluid dynamics conference*, 2013.

[Fan17] Duoming Fan. *On the acoustic component of active flux schemes for nonlinear hyperbolic conservation laws*. PhD thesis, University of Michigan, Dissertation, 2017.

[vL77] Bram van Leer. Towards the ultimate conservative difference scheme. IV. A new approach to numerical convection. *Journal of computational physics*, 23(3):276–299, 1977.

[vL79] Bram van Leer. Towards the ultimate conservative difference scheme. v. A second-order sequel to Godunov's method. *Journal of computational Physics*, 32(1):101–136, 1979.

STRUCTURAL SCHEMES FOR HAMILTONIAN SYSTEMS

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ABSTRACT

We present an adaptation of the so-called structural method [1] for Hamiltonian systems [2], and redesign the method for this specific context, which involves two coupled differential systems. Structural schemes decompose the problem into two sets of equations: the physical equations, which describe the local dynamics of the system, and the structural equations, which only involve the discretization on a very compact stencil. After a general description of the scheme for the scalar case we extending the technique to the vector case (treating e.g. the n-body system). The scheme is also addapted to non-separable system (e.g. a charged particle in an electromagnetic field). We give numerical evidence of the method's efficiency, its capacity to preserve invariant quantities such as the total energy, and draw comparisons with the traditional symplectic methods.

INTRODUCTION

Hamiltonian systems, a class of ordinary differential equations (ODEs) that arise from Hamiltonian mechanics, play a fundamental role in the mathematical modeling of physical systems with conserved quantities, such as total energy for a closed system. Due to their rich structure, the numerical approximation of Hamiltonian systems presents several challenges. One of the main issues is the preservation of geometric features, such as symplecticity and conserved quantities like energy or angular momentum, which are integral to the physical behavior of the system. Standard numerical methods, even of very high order, such as explicit or implicit Runge-Kutta schemes, often fail to maintain these properties during simulations, generating error accumulations in the long run that can become critical and totally spoil the quality of the solution [3].

The structural method was recently introduced, in [1], to solve ODE systems. It provides a systematic way of constructing implicit schemes with a highly compact stencil, which have an arbitrarily high order of accuracy together with unconditional stability. We develop and adapt the structural method to Hamiltonian systems [2], taking into account the specificities of such systems, namely the conservation of several invariants, and the two-variable nature of the problem.

A SHORT OVERVIEW OF THE ZD STRUCTURAL METHOD

We provide a short review of the structural method based on the so-called Physical and Structural Equations. Let us consider the Ordinary Differential Equation (ODE) for $t \in [0,T]$

$$\dot{x} = f(x), \qquad x(0) = x_0. \tag{1}$$

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Traditional schemes blend the discretization with the physical equation, that is, the function that describe the dynamic of the physical system. For example, the popular Crank-Nicholson scheme reads

$$\frac{x_{n+1}-x_n}{\Delta t}=\frac{f(x_n)+f(x_{n+1})}{2}$$

where the left-hand side is the time derivative discretization, while the right-hand side represents the physics (i.e., the function f that characterizes the physical problem). Obviously, one can split the scheme into two equations, namely

$$D_{n+1}=f(Z_{n+1}) \qquad \text{and} \qquad \frac{Z_{n+1}-Z_n}{\Delta t}-\frac{D_n+D_{n+1}}{2}=0,$$

where Z_n and D_n are approximations of the Zeroth-order derivative $x(t_n)$ and the first-order Derivative $\dot{x}(t_n)$

Following the example, the idea of structural method consists in splitting the problem with, on the one hand, the Physical Equations `PE` and, and the other hand, the Structural Equations `SE`. The two sets of equations involve the unknowns over a block of size R steps corresponding to the time step t_{n+1} until t_{n+R}, given the initial configuration at the time t_n. We obtain a nonlinear system combining the function approximations together with the derivatives for all the intermediate steps.

The structural method

The simple version called the `ZD` scheme, where one only uses implicit combinations of the approximate function and first derivative as unknowns. The generic `ZD{` structural equation then reads

$$\sum_{r=0}^{R} a_{r,0}Z_{n+r}+a_{r,1}D_{n+r}=0, \tag{2}$$

where $(a_{r,s})_{r\in\{0,\dots,R\},s\in\{0,1\}}$ are the coefficients of the `SE`, independent of n by assuming a uniform time discretization with time step Δt. It is important to remark that $r=0$ corresponds to the time t_n where all the variables are given. In total, $2(R+1)$ coefficients characterize the structural equation. They can be reshaped as a vector

$$a=\left[a_{0,0},\quad a_{1,0},\quad \dots,\quad a_{R,0},\quad a_{0,1},\quad a_{1,1},\quad \dots,\quad a_{R,1}\right].$$

On the other hand, the implicit problem involves $2R$ unknowns with R physical equations `PE[1]`, ..., `PE[R]` corresponding to the relations

$$D_{n+r}=f(Z_{n+r}), \qquad \text{for all } r\in\{1,\dots,R\}.$$

Hence, the structural method requires R linearly independent structural equations, i.e. R vectors, $(a^m)_{m\in\{1,\dots,R\}}$ whose components are denoted by $(a^m_{r,s})_{r,s,m}$, to close the system of size $2R$.

Determining the Structural equations.

To provide this set of R structural equations ensuring that the resulting scheme is high-order accurate, we define the functional

$$E(a,\phi)=\sum_{r=0}^{R} a_{r,0}\phi(t_r)+a_{r,1}\phi'(t_r). \tag{3}$$

Taking the particular case of polynomial functions $\pi_\ell(t)=t^{\ell-1}$ for $\ell\in\{1,\dots,2(R+1)\}$, we consider the linear system $E(a,\pi_\ell)=0$, which rewrites as the $2(R+1)$ equations

$$\forall\ell\in\{1,\dots,2(R+1)\},\quad E(a;\pi_\ell)=\sum_{r=0}^{R} a_{r,0}\pi_\ell(r\Delta t)+a_{r,1}\pi'_\ell(r\Delta t)=0.$$

We obtain a $2(R+1)\times 2(R+1)$ non-singular linear system given by $Ma=0$, where the matrix M gathers all the coefficients $\pi_\ell(r\Delta t)$ and $\pi'_\ell(r\Delta t)$, for all $\ell\in\{1,\dots,2(R+1)\}$ and $r\in\{0,\dots,R\}$. A set of R structural equations is given by an orthogonormal basis of the Kernel and one has the relations

$$\sum_{r=0}^{R} a^m_{r,0}Z_{n+r}+a^m_{r,1}D_{n+r}=0, \quad m=1,\cdots,R, \tag{4}$$

exact for polynomials up to degree $R+1$.

EXTENSION TO A SCALAR HMAILTONIAN SYSTEM

Consider the hamiltonian $\mathscr{H}(x,p)$ in function of position x and momentum p. We define the trajectories as the solution of the ODEs `PE`

$$\dot{x} = \partial_p \mathscr{H}(x,p) \quad \text{and} \quad \dot{p} = -\partial_x \mathscr{H}(x,p),$$

with the initial condition $x(0) = x_0$, $p(0) = p_0$ or using the other notation

$$D\mathrm{x} = \partial_p \mathscr{H}(\mathbb{Z}\mathrm{x}, \mathbb{Z}\mathrm{p}), \tag{5}$$

$$D\mathrm{p} = -\partial_x \mathscr{H}(\mathbb{Z}\mathrm{x}, \mathbb{Z}\mathrm{p}). \tag{6}$$

Approximations of the function x and the derivative $\dot{x}$ are connected via the structural equations (4), and similarly for the function p and its derivative.

Denoting by $\mathbb{Z}\mathrm{x}_{n+r}$ and $\mathbb{Z}\mathrm{p}_{n+r}$ the approximation of $x(t_{n+r})$ and $p(t_{n+r})$ respectively (and the same notations for $D\mathrm{x}_{n+r}$ and $D\mathrm{p}_{n+r}$ for the derivative), the structural equations read

$$\forall m \in \{1,\ldots,R\}, \qquad \begin{cases} \displaystyle\sum_{r=0}^{R} a_{r,0}^{m} \mathbb{Z}\mathrm{x}_{n+r} + a_{r,1}^{m} D\mathrm{x}_{n+r} = 0, \\ \displaystyle\sum_{r=0}^{R} a_{r,0}^{m} \mathbb{Z}\mathrm{p}_{n+r} + a_{r,1}^{m} D\mathrm{p}_{n+r} = 0. \end{cases}$$

It is important to note that the x and p use the **same** structural equations (the same coefficients $a_{r,s}^{m}$) and **only differ** through the physical ones. Let denote by

$$\flat\mathbb{Z}\mathrm{x}_n = \left(\mathbb{Z}\mathrm{x}_{n+1}, \mathbb{Z}\mathrm{x}_{n+2}, \ldots, \mathbb{Z}\mathrm{x}_{n+R}\right)^{\mathrm{t}} \qquad \text{and} \qquad \flat\mathbb{Z}\mathrm{p}_n = \left(\mathbb{Z}\mathrm{p}_{n+1}, \mathbb{Z}\mathrm{p}_{n+2}, \ldots, \mathbb{Z}\mathrm{p}_{n+R}\right)^{\mathrm{t}},$$

the respective R-block vectors for the functions x and p. Similarly, the R-block vectors for the first derivatives are denoted by $\flat D\mathrm{x}_n$ and $\flat D\mathrm{p}_n$. The structural equations for the `ZD` scheme of size R then read

$$0 = \flat\mathbb{Z}\mathrm{x}_n + B_d\, \flat D\mathrm{x}_n + \mathbb{Z}\mathrm{x}_n\, b_z + D\mathrm{x}_n\, b_d, \tag{7}$$

$$0 = \flat\mathbb{Z}\mathrm{p}_n + B_d\, \flat D\mathrm{p}_n + \mathbb{Z}\mathrm{p}_n\, b_z + D\mathrm{p}_n\, b_d. \tag{8}$$

with B_d, b_z, b_d the matrix and vectors evaluated from the coefficients of the R structural equations.

Given the approximation at time t_r, we produce two sequences $\left(\flat\mathbb{Z}\mathrm{x}_n^{(k)}, \flat D\mathrm{x}_n^{(k)}\right)$ and $\left(\flat\mathbb{Z}\mathrm{p}_n^{(k)}, \flat D\mathrm{p}_n^{(k)}\right)$ that converge to the solution for the next R times step. The fixed-point algorithm is then given by the following iterative procedure.

- **Initialization.** To build $\flat\mathbb{Z}\mathrm{x}_n^{(0)}$, $\flat D\mathrm{x}_n^{(0)}$, $\flat\mathbb{Z}\mathrm{p}_n^{(0)}$, $\flat D\mathrm{p}_n^{(0)}$, we set for $r \in \{1,\ldots,R\}$

$$\mathbb{Z}\mathrm{x}_{n+r}^{(0)} = \mathbb{Z}\mathrm{x}_{n+r-1}^{(0)} + \Delta t\, D\mathrm{x}_{n+r-1}^{(0)},$$
$$\mathbb{Z}\mathrm{p}_{n+r}^{(0)} = \mathbb{Z}\mathrm{p}_{n+r-1}^{(0)} + \Delta t\, D\mathrm{p}_{n+r-1}^{(0)},$$
$$D\mathrm{x}_{n+r}^{(0)} = \partial_p \mathscr{H}\left(\mathbb{Z}\mathrm{x}_{n+r}^{(0)}, \mathbb{Z}\mathrm{p}_{n+r}^{(0)}\right),$$
$$D\mathrm{p}_{n+r}^{(0)} = -\partial_x \mathscr{H}\left(\mathbb{Z}\mathrm{x}_{n+r}^{(0)}, \mathbb{Z}\mathrm{p}_{n+r}^{(0)}\right),$$

  with $\mathbb{Z}\mathrm{x}_n^{(0)} = \mathbb{Z}\mathrm{x}_n$, $D\mathrm{x}_n^{(0)} = D\mathrm{x}_n$ and $\mathbb{Z}\mathrm{p}_n^{(0)} = \mathbb{Z}\mathrm{p}_n$, $D\mathrm{p}_n^{(0)} = D\mathrm{p}_n$.

- **Iteration.** We first compute a new approximation for the solution using the structural equations

$$\flat\mathbb{Z}\mathrm{x}_n^{(k+1)} = -\left(\mathbb{Z}\mathrm{x}_n\, b_z + D\mathrm{x}_n\, b_d + B_d\, \flat D\mathrm{x}_n^{(k)}\right),$$
$$\flat\mathbb{Z}\mathrm{p}_n^{(k+1)} = -\left(\mathbb{Z}\mathrm{p}_n\, b_z + D\mathrm{p}_n\, b_d + B_d\, \flat D\mathrm{p}_n^{(k)}\right),$$

  and then update the first derivatives with the physical equations deriving from the Hamiltonian

$$D\mathrm{x}_{n+r}^{(k+1)} = \partial_p \mathscr{H}\left(\mathbb{Z}\mathrm{x}_{n+r}^{(k+1)}, \mathbb{Z}\mathrm{p}_{n+r}^{(k+1)}\right),$$
$$D\mathrm{p}_{n+r}^{(k+1)} = -\partial_x \mathscr{H}\left(\mathbb{Z}\mathrm{x}_{n+r}^{(k+1)}, \mathbb{Z}\mathrm{p}_{n+r}^{(k+1)}\right).$$

- **Stopping criterion.** We end the fixed-point when two successive solutions are close enough according to the tolerance parameter `tol`, that is $\|\flat\mathbb{Z}\mathrm{x}^{(k+1)} - \flat\mathbb{Z}\mathrm{x}^{(k)}\| \leq \mathtt{tol}$.

A NUMERICAL EXAMPLE

the one-dimensional pendulum system [4] is governed by the Hamiltonian

$$\mathscr{H}(x,p) = \frac{p^2}{2m\ell^2} + mg\ell(1-\cos(x)) \tag{9}$$

where m is the mass, ℓ the length of the pendulum and g the gravity. The differential system deriving from Hamilton's equations reads

$$\dot{x} = \frac{p}{m}, \qquad \dot{p} = -mg\ell\sin(x) \tag{10}$$

and the physical equations `PE` is given by

$$D\mathrm{x} = Z\mathrm{p}/m, \qquad D\mathrm{p} = -mg\ell\sin(Z\mathrm{x}).$$

The numerical applications have been carried out with $m=1$, $g=1$, $\ell=1$, and the initial conditions $x(0)=\pi/4$ and $p(0)=0$. In [4], an analytical solution is derived, whose expression, not reported here, involves a Jacobi elliptic function. The exact position and momentum at final time $T=100$ are given by

$$x(t=100) = -0.2633498226088722 \qquad \text{and} \qquad p(t=100) = -0.7189111241830892$$

in `double` precision (since the Jacobi functions are not implemented with arbitrary precision). Hence, schemes with very high accuracy will reach machine error with a comparatively low number of points.

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REFERENCES

[1] S. Clain, G. J. Machado, M. T. Malheiro *Compact schemes in time with applications to partial differential equations* Comput. Math. Appl., vol 140, 2023.

[2] S. Clain, E. Franck, V. Michel-Dansac *Structural schemes for hamiltonian systems* arXiv 2501.13515, 2025. https://arxiv.org/abs/2501.13515

[3] H. Beust *Symplectic integration of hierarchical stellar systems* Astron. Astrophys., vol 400, Num 3 2003.

[4] A. Beléndez and C. Pascual and D.I. Méndez and T. Beléndez and C. Neipp *Exact solution for the nonlinear pendulum* Rev. Bras. Ens. Fis., vol 29, num 4, 2007.

TABLE 1: Pendulum system: error on the position at $T = 100$s.

(a) Errors obtained with the ZD scheme.

	R=2		R=4		R=6		R=8	
N	ex	ordx	ex	ordx	ex	ordx	ex	ordx
120	8.27e-01	—	3.32e-01	—	3.25e-02	—	—	—
240	3.36e-02	4.6	1.11e-02	4.9	2.24e-04	7.2	1.32e-02	—
480	2.45e-03	3.8	5.56e-05	7.6	4.85e-06	5.5	6.89e-06	10.9
960	1.58e-04	4.0	8.81e-07	6.0	1.05e-08	8.8	4.92e-09	10.5
1920	9.80e-06	4.0	1.43e-08	5.9	4.27e-11	7.9	7.98e-13	12.6

(b) Errors obtained with the ZDS scheme.

	R=1		R=2		R=3		R=4	
N	ex	ordx	ex	ordx	ex	ordx	ex	ordx
120	3.93e-02	—	1.26e-02	—	2.00e-03	—	—	—
240	2.63e-03	3.9	2.16e-05	9.2	6.24e-06	8.3	1.58e-05	—
480	1.66e-04	4.0	4.35e-07	5.6	1.94e-09	11.7	3.32e-09	12.2
960	1.04e-05	4.0	6.93e-09	6.0	6.25e-12	8.3	1.21e-13	14.7
1920	6.52e-07	4.0	1.09e-10	6.0	4.02e-14	7.3	***	***

(c) Errors obtained with the classical schemes.

	MA2		CS4		KL6		KL8	
N	ex	ordx	ex	ordx	ex	ordx	ex	ordx
120	8.60e-01	—	3.26e-04	—	3.51e-03	—	6.71e-06	—
240	2.20e-01	2.0	3.89e-05	3.1	6.16e-05	5.8	1.98e-08	8.4
480	5.47e-02	2.0	4.84e-06	3.0	9.75e-07	6.0	7.62e-11	8.0
960	1.36e-02	2.0	6.05e-07	3.0	1.53e-08	6.0	3.02e-13	8.0
1920	3.41e-03	2.0	7.59e-08	3.0	2.40e-10	6.0	***	***

TABLE 2: Pendulum system: error on the Hamiltonian at T=100s.

(a) Errors obtained with the `ZD` scheme.

	R=2		R=4		R=6		R=8	
N	eH	ordH	eH	ordH	eH	ordH	eH	ordH
120	1.66e-01	—	2.00e-01	—	5.30e-02	—	—	—
240	2.10e-02	3.0	6.23e-03	5.0	6.12e-04	6.4	8.57e-03	—
480	1.49e-03	3.8	6.71e-06	6.5	3.78e-06	7.3	2.60e-06	11.
960	9.56e-05	4.0	9.98e-08	6.0	6.52e-09	9.2	3.30e-09	9.6
1920	6.00e-06	4.0	1.51e-09	6.0	2.34e-11	8.1	3.91e-13	9.7

(b) Errors obtained with the `ZDS` scheme.

	R=1		R=2		R=3		R=4	
N	eH	ordH	eH	ordH	eH	ordH	eH	ordH
120	1.87e-03	—	9.90e-03	—	9.41e-04	—	—	—
240	1.15e-04	4.0	1.57e-05	9.3	3.96e-06	7.9	1.20e-05	—
480	7.21e-06	4.0	2.17e-07	6.1	9.55e-09	8.7	2.24e-09	12.
960	4.51e-07	4.0	3.33e-09	6.0	3.25e-11	8.2	5.14e-13	12.
1920	2.82e-08	4.0	5.18e-11	6.0	1.24e-13	8.0	4.60e-16	10.

(c) Errors obtained with the classical schemes.

	MA2		CS4		KL6		KL8	
N	ex	ordx	ex	ordx	ex	ordx	ex	ordx
120	1.40e-02	—	4.61e-04	—	4.76e-05	—	1.36e-06	—
240	1.71e-03	3.0	4.24e-05	3.4	5.59e-07	6.4	4.26e-09	8.3
480	2.45e-04	2.8	4.63e-06	3.2	8.15e-09	6.1	1.61e-11	8.0
960	4.20e-05	2.5	5.41e-07	3.1	1.25e-10	6.0	6.26e-14	8.0
1920	8.74e-06	2.3	6.54e-08	3.0	1.95e-12	6.0	2.44e-16	8.0

MINIMALLY IMPLICIT METHODS FOR HYPERBOLIC EQUATIONS WITH STIFF SOURCE TERMS

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ABSTRACT

I will present the Minimally-Implicit Runge Kutta (MIRK) methods for the numerical resolution of hyperbolic equations with stiff source terms. These methods have been successfully applied to the resistive relativistic magnetohydrodynamic (RRMHD) equations [1] and the M1 neutrino transport equations [2]. Previous approaches in these equations rely on using the so-called Implicit-Explicit (IMEX) Runge-Kutta schemes [3]; instead, MIRK methods are able to deal with stiff terms producing stable numerical evolutions and their computational cost is similar to the standard explicit methods. I will show some initial steps towards applying the MIRK methods to the shallow water equations with friction. The aim of this talk is also to find potential new applications of these methods to other examples in geophysical contexts.

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REFERENCES

[1] I. Cordero-Carrión, S. Santos-Pérez, C. Martínez-Vidallach *Numerical evolution of the resistive relativistic magnetohydrodynamic equations: A minimally implicit Runge-Kutta scheme* Appl. Maths. Comp., vol 443, 127774, 2023.

[2] S. Santos-Pérez, M. Obergaulinger, I. Cordero-Carrión *Minimally implicit methods for the numerical integration of the neutrino transport equations* arXiv:2302.12089, 2023.

[3] L. Pareschi, G. Russo *Implicit-explicit Runge-Kutta schemes and applications to hyperbolic systems with relaxation* J. Sci. Comp., 25, 129–155, 2005.

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A CERTIFIED POD-GREEDY METHOD BASED ON LEGENDRE COLLOCATION FOR A RAYLEIGH-BÉNARD PROBLEM

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ABSTRACT

This work presents a certified reduced basis method for studying bifurcations in a Rayleigh-Bénard problem using a Legendre collocation spectral method as high-fidelity discretization. The method is designed to handle independently each branch of the bifurcation diagram, producing for each one a certified reduced-order model that is formulated as a least-squares problem minimizing the restriction of the high-fidelity residual to a reduced basis. To certify the method, the different reduced bases are constructed iteratively through a POD-greedy algorithm driven by rigorous a posteriori error estimates, which are defined as the quotient of a high-fidelity residual and a stability factor. A notable contribution of this work is the use of another reduced basis approach to approximate the stability factor. Furthermore, the proposed methodology does not require prior knowledge of the location of bifurcation points, accurately locating them in the online stage using reduced-order indicators. Overall, this method allows for accurate, fast, and reliable computation of the bifurcation diagram of the Rayleigh-Bénard problem.

INTRODUCTION

In this study, a reduced-order method is applied to a 2D Rayleigh-Bénard bifurcation problem consisting in the continuity, Navier-Stokes, and heat equations, and depending on a parameter, the Rayleigh number R [1].

The proposed reduced-order method is a surrogate of Legendre collocation, hereinafter referred to as high-fidelity scheme, and belongs to the reduced basis (RB) paradigm [2]. In particular, its implementation relies on two key ingredients: the computation of a low-dimensional basis, referred to as reduced basis, and the definition of a reduced-order model (ROM), which is an algebraic problem with a reduced number of degrees of freedom whose solutions approximate the solutions of the high-fidelity scheme. As other RB methods, the proposed methodology works in two stages. The first stage, called offline stage, involves some time-consuming computations targeted to enable a second stage, the online stage, designed to be fast and computationally efficient.

The proposed RB method handles independently each branch of the bifurcation diagram, producing, for each one, a ROM based on a different reduced basis. The method is certified because, in every branch, by using a training loop that updates iteratively the reduced basis accordingly to an a posteriori error estimate, we can ensure that the errors between the solutions produced by the ROM and the corresponding high-fidelity solutions are below a given threshold.

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PROBLEM FORMULATION

The governing equations of the 2D Rayleigh-Bénard convection process under study are

$$\nabla \cdot v = 0, \qquad \text{in } \Omega, \tag{1a}$$

$$\theta e_z - \nabla p + \frac{1}{\sqrt{R}} \Delta v = 0, \qquad \text{in } \Omega, \tag{1b}$$

$$v \cdot \nabla \theta - w - \frac{1}{\sqrt{R}} \Delta \theta = 0. \qquad \text{in } \Omega, \tag{1c}$$

Here, $\Omega \subset \mathbb{R}$ is a rectangular domain; v, θ and p are, respectively, the velocity, temperature and pressure fields of the fluid; e_z denotes the upward vertical unitary vector; R is an adimensional number known as Rayleigh number; and ∇ and Δ denote the standard nabla and Laplace operators in Cartesian coordinates. The unknowns are the velocity, temperature and pressure field of the fluid and R is the parameter studied.

THE REDUCED-ORDER MODEL

The ROM is defined as follows. Given the reduced basis matrix $\Psi \in \mathbb{R}^{N_h \times N}$ (here, N_h denotes de number of degrees of freedom of the high-fidelity discretization, and N the dimension of the reduced space), the reduced-order solution corresponding to the parameter $R = \widetilde{R}$ is a vector $\widetilde{c} \in \mathbb{R}^N$ such that,

$$\widetilde{c} := \arg\min_{c \in \mathbb{R}^N} \left\| G(\Psi c; \widetilde{R}) \right\|_2^2, \tag{2}$$

where $G(\cdot;\cdot) : \mathbb{R}^{N_h} \times \mathbb{R} \to \mathbb{R}^{N_h}$ is the residual of the high-fidelity discretization, and $\|\cdot\|_2$ denotes the Euclidean norm in $\mathbb{R}^{N_h}$. The expression (2) defines a non-linear least-squares problem whose solution is approximated numerically using a standard Gauss-Newton iteration.

SOME RESULTS

With the proposed methodology, the target bifurcation diagram can be recovered with high precision and reduced computational cost. Fig. 1 shows a comparison of the high-fidelity bifurcation diagram and the reduced-order counterpart.

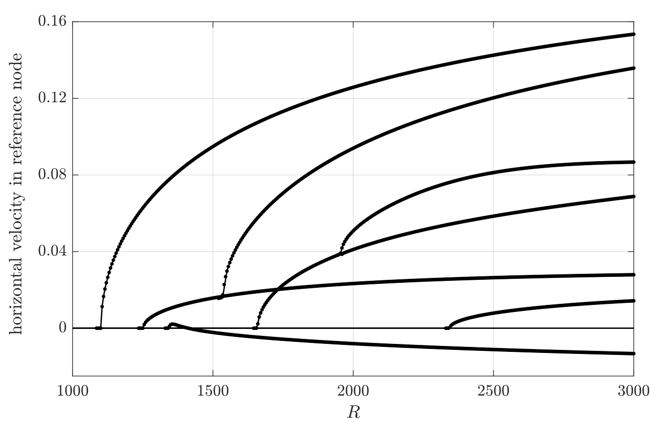

FIGURE 1: Bifurcation diagram computed using the proposed reduced-order method. Dots represent reduced solutions. Lines denote high-fidelity solutions.

REFERENCES

[1] Getling, A.V. *Rayleigh-Bénard Convection: Structures and Dynamics*. World Scientific, 1998.

[2] Quarteroni, A., Manzoni, A., and Negri, F. *Reduced Basis Methods for Partial Differential Equations: An Introduction* Springer Cham, Switzerland, 2016.

VERY HIGH-ORDER ACCURATE FINITE VOLUME SCHEMES FOR VORTICITY-BASED FORMULATIONS OF THE INCOMPRESSIBLE NAVIER-STOKES EQUATIONS

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ABSTRACT

The Navier-Stokes equations, based on pressure and velocity variables, are the conventional formulation for simulating incompressible fluid flow problems. However, the pressure-velocity coupling and the absence of pressure boundary conditions pose challenging numerical problems, which researchers have tried to overcome by developing a variety of numerical techniques. One approach is to rewrite the momentum balance equation in terms of the vorticity vector, which reduces to a scalar transport equation for two-dimensional flows. Additional variables and equations are then complemented to the vorticity transport equation to enforce the incompressibility constraint, as in the streamfunction-vorticity or velocity-vorticity formulations. Compared to the primitive formulation, these vorticity-based formulations do not require pressure to be calculated, thus avoiding the inherent difficulties associated with pressure-velocity coupling and the absence of pressure boundary conditions. The present work proposes a simple, efficient, high-order accurate finite volume discretisation based on the reconstruction for off-site data (ROD) method [1] for the solution of the two-dimensional incompressible Navier-Stokes equations in the streamfunction-vorticity formulation. A comprehensive discussion is devoted to several key aspects in the numerical solution of vorticity-based formulations, such as vorticity boundary conditions and multiply connected domains. Several benchmark test cases of incompressible fluid flow problems in non-trivial two-dimensional curved domains are addressed, and the proposed method effectively achieves convergence up to the sixth-order.

REFERENCES

[1] R. Costa, S. Clain, G.J. Machado, J.M. Nóbrega, H. Beirão da Veiga, and F. Crispo, Imposing general slip conditions on curved boundaries for 3D incompressible flows with a very high-order accurate finite volume scheme on polygonal meshes, Computer Methods in Applied Mechanics and Engineering 415, 116274, 2023. DOI: https://doi.org/10.1016/j.cma.2023.116274.

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LAYER-AVERAGED MODELS FOR COMPLEX RHEOLOGIES

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ABSTRACT

In this talk we present a serie of laver-averaged models that approximates Navier-Stokes system with a complex rheology (see [1], [2], [3], [4]). The simpler ones correspond to models with hydrostatic pressure and the main order terms of the stress tensor component by introducing an asymptotic approximation. The complex ones are based in the models introduced in [5] with non-hydrostatic pressure and all components of the stress tensor. Intermediate models will be also presented. Finally, we consider applications for the case of Herschel-Bulkley and mu(I)-rheology models.

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REFERENCES

[1] E. Audusse, M. Bristeau, B. Perthame, J. Sainte-Marie. *A multilayer Saint-Venant system with mass exchanges for shallow water flows. derivation and numerical validation.* ESAIM: Mathematical Modelling and Numerical Analysis, 45:169–200, 2011.

[2] M.O. Bristeau, C. Guichard, B. Di Martino, J. Sainte-Marie. *Layer-averaged Euler and Navier-Stokes equations*. Communications in Mathematical Sciences, 15(5):1221–1246, 2017. COMMUN. MATH. SCI. 2017 International Press

[3] C. Escalante, E.D. Fernández-Nieto, J Garres-Díaz, A. Mangeney. *Multilayer Shallow Model for Dry Granular Flows with a Weakly Non-hydrostatic Pressure.* Journal of Scientific Computing, 96(3), 88, 2023.

[4] E.D. Fernández-Nieto, J Garres-Díaz, P. Vigneaux. *Multilayer models for hydrostatic Herschel-Bulkley viscoplastic flows.* Computers & Mathematics with Applications, 139: 99–117, 2023.

[5] E.D. Fernández-Nieto, J Garres-Díaz. *Layer-averaged approximation of Navier-Stokes system with complex rheologies.* ESAIM: Mathematical Modelling and Numerical Analysis, 57(5):2735–2774, 2023.

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A GEOMETRICAL GREEN-NAGHDI TYPE SYSTEM FOR FLUID FLOWS IN CHANNELS

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ABSTRACT

We consider 2D free-surface gravity waves described by the Saint-Venant equations with periodic bathymetric variation transverse to a given direction. Averaged in the transverse direction, the corresponding 1D equations represent a fully non-linear Galilean invariant model admitting a variational formulation under a natural assumption concerning the symmetry of the periodic element of the bottom topography. In particular, we show that the symmetry assumption allows the model to be used for flows in prismatic channels. The system is recast in two useful forms appropriate for its numerical approximations. Numerical results validate our model against 2D non-linear shallow water simulations.

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POD-BASED WELL-BALANCED REDUCED-ORDER MODELS FOR HYPERBOLIC PDE SYSTEMS IN THE FINITE VOLUME FRAMEWORK

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ABSTRACT

This work presents a reduced-order modeling (ROM) approach for hyperbolic balance laws, integrating Proper Orthogonal Decomposition (POD) with the Discrete Empirical Interpolation Method (DEIM) and Proper Interval Decomposition (PID). The methodology is designed to preserve well-balanced properties in the reduced models, ensuring accurate approximation of stationary solutions. We apply this framework to various test cases, including the transport equation with source term, the non-homogeneous Burgers equation, and the shallow water equations with variable bathymetry and Manning friction.

A theoretical analysis establishes the advantages of employing well-balanced full-order models (FOMs) as a basis for ROMs. Additionally, we extend our approach to parameter-dependent systems, demonstrating its capability to provide efficient and robust approximations across different parameter values. Numerical experiments validate the effectiveness of the proposed technique.

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ASYMPTOTICALLY WELL-BALANCED GEOSTROPHIC RECONSTRUCTION FOR THE 2D RSWE IN SPHERICAL COORDINATES. LINEAR DISPERSION RELATION ANALYSIS.

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ABSTRACT

The dynamics of large-scale geophysical fluids is primarily governed by the balance between the Coriolis force and the pressure gradient. This phenomenon, known as geostrophic equilibrium, is the basis for the geostrophic model, which has proven to be extremely useful for understanding and forecasting large-scale atmospheric and oceanic dynamics. In the present work, we develop second- and third-order finite-volume numerical schemes applied to the 2D rotating shallow-water equations in spherical coordinates. These schemes are designed to preserve the geostrophic equilibrium in the limit as the Rossby number tends to zero. The final goal is to design reliable and efficient forecasting models for simulating meteotsunamis, long-wave events generated in the ocean by atmospheric pressure disturbances. These disturbances produce long waves of small amplitude that gradually amplify as they approach the coast. The numerical results for various analytical and real-world test cases underscore the importance of maintaining geostrophic equilibrium over time. An additional analysis of the linear dispersive relation in the numerical schemes is carried out by comparing the effects of TVD Runge-Kutta time integrators with symplectic integrators, exploiting the Hamiltonian structure of the system induced by the Coriolis force.

INTRODUCTION

Trans-oceanic long waves can be modelled using the 2D Shallow Water Equations in spherical coordinates. The Coriolis force plays a major role when simulating these kind of long-lasting tsunami-like waves, as its influence may provoke a convergence into a stationary state known as geostrophic equilibrium. The geostrophic equilibrium is a steady state that occurs when the Coriolis force finds balance with the pressure gradient. It is well-known that, at large scales, geophysical flows are often perturbations of the stationary solution. Second and third order finite volume numerical schemes have been developed for the 2D SWE in spherical coordinates using MUSCL and CWENO spatial reconstructor operators together with a TVD Runge-Kutta time integrator. The main novelty in these numerical schemes is the asymptotic preservation of the spherical version of the quasi-geostrophic equilibrium –a more handy version of the geostrophic equilibrium maintaining the core properties. The quasi-geostrophic equilibrium can be derived linearizing the equations with a flat bottom around a background still state. At large scales, the small perturbations ε are of the order of the Froude and Rossby numbers $\varepsilon \approx 10^{-2}$. By reconstructing a local cuadratic (second order scheme) or cubic (third order scheme) free surface whose gradient is in balanced with the local Coriolis force, we prove that the schemes are well-balanced as $\varepsilon \to 0$, with order $O(\varepsilon^2)$.

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ASIMPTOTICALLY WELL-BALANCED SCHEMES

The shallow water system on the sphere, accounting for bottom topography, the Coriolis force, and non-constant, varying in time and space, atmospheric source terms, can be expressed as a system of balance laws [1, 2] of the form:

$$\begin{cases} \partial_t h_\sigma + \frac{1}{R}\left[\partial_\theta\left(\frac{Q_\theta}{\sigma}\right) + \partial_\varphi Q_\varphi\right] = 0, \\ \partial_t Q_\theta + \frac{1}{R}\partial_\theta\left(\frac{Q_\theta^2}{h_\sigma \sigma}\right) + \frac{1}{R}\partial_\varphi\left(\frac{Q_\theta Q_\varphi}{h_\sigma}\right) + \frac{Q_\theta Q_\varphi}{R h_\sigma \sigma}\partial_\varphi \sigma \\ \qquad + \frac{g h_\sigma}{R\sigma^2}\partial_\theta \eta_\sigma = f Q_\varphi - \frac{h_\sigma}{\rho R \sigma}\partial_\theta p^a, \\ \partial_t Q_\varphi + \frac{1}{R}\partial_\theta\left(\frac{Q_\theta Q_\varphi}{h_\sigma \sigma}\right) + \frac{1}{R}\partial_\varphi\left(\frac{Q_\varphi^2}{h_\sigma}\right) - \left(\frac{Q_\theta^2}{R h_\sigma \sigma} + \frac{g h_\sigma \eta_\sigma}{R\sigma^2}\right)\partial_\varphi \sigma \\ \qquad + \frac{g h_\sigma}{R\sigma}\partial_\varphi \eta_\sigma = -f Q_\theta - \frac{h_\sigma}{\rho R}\partial_\varphi p^a, \end{cases} \tag{1}$$

Second- and third-order path-conservative finite volumes numerical schemes are developed for this PDE system, employing reconstruction operators. Moreover, these reconstruction operators integrate a high-order approximation of the geostrophic equilibrium structure:

$$\begin{cases} \partial_\theta u_\theta + \partial_\varphi(\sigma u_\varphi) = 0, \\ g\partial_\theta \eta = R f \sigma u_\varphi, \\ g\partial_\varphi \eta = -R f u_\theta, \end{cases} \tag{2}$$

where $\eta = h - H$, $\sigma = \cos\varphi$ and $f = 2\Omega \sin\varphi$.

Following this procedure, the inclusion of high-order approximations of (2) leads to numerical schemes that asymptotically preserve this steady state, yielding improved numerical results in situations with low Froude and Rossby numbers.

LINEAR DISPERSION RELATION

The inclusion of the Coriolis force in the SWE system gives rise to a dispersive PDE system, where the phase velocity depends on the wave number. The linearized form of (1) around a constant background state $h = h_0$, $u_\theta = 0$, $u_\varphi = 0$ with flat bottom is given by:

$$\begin{cases} \partial_t h + \frac{1}{R\sigma^2}\partial_\theta Q_\theta + \frac{1}{R\sigma}\partial_\varphi Q_\varphi = 0, \\ \partial_t Q_\theta + \frac{g h_{0,\sigma}}{R\sigma}\partial_\theta h - f Q_\varphi = 0 \\ \partial_t Q_\varphi + \frac{g h_{0,\sigma}}{R}\partial_\varphi h + f Q_\theta = 0 \end{cases} \tag{3}$$

Seeking solutions of system (3) in the form:

$$\begin{aligned} h &= \widetilde{h}\, e^{i(k_\theta \theta + k_\varphi \varphi - \omega t)} \\ Q_\theta &= \widetilde{Q_\theta}\, e^{i(k_\theta \theta + k_\varphi \varphi - \omega t)} \\ Q_\varphi &= \widetilde{Q_\varphi}\, e^{i(k_\theta \theta + k_\varphi \varphi - \omega t)} \end{aligned} \tag{4}$$

leads to the dispersion relation:

$$\omega^2 = f^2 + \frac{g h_0}{R^2}\left(\frac{k_\theta^2}{\cos^2\theta} + k_\varphi^2\right)$$

The linear dispersion relation in the developed asymptotic well-balanced schemes is compared to the exact one, presenting different strategies to improve its accuracy and illustrating the amplification of these harmonic waves.

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REFERENCES

[1] Tort, M. & Dubos, T. & Bouchut, F. & Zeitlin, V. *Consistent shallow-water equations on the rotating sphere with complete Coriolis force and topography. Journal of Fluid Mechanics. 748. 789-821., 2014.*

[2] Castro, M & Ortega, S. & Madroñal, C.. *Well-balanced methods for the shallow water equations in spherical coordinates. Computers & Fluids. 169., 2018.*

[3] Arakawa, K., Vivian R. Lamb *Computational Design of the Basic Dynamical Processes of the UCLA General Circulation Model, vol 17, Elsebeer, 1977.*

A QUASI SECOND ORDER LOW MACH PRESSURE CORRECTION SCHEME FOR COMPRESSIBLE FLOWS

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ABSTRACT

In this work, we present a low Mach number staggered pressure correction scheme with algebraic MUSCL (A-MUSCL) convection fluxes [2, 3] for the compressible Euler and Navier-Stokes equations. This solver is to be used by the Institute of Radioprotection and Nuclear Safety (IRSN) for deflagrations simulations. It has been implemented in the IRSN open-source code CALIF^{3}S.

In a first step, a tentative velocity is computed using the momentum balance equation, where the beginning-of-step value of the pressure is used. Then, a correction equation is introduced so that the end-of-step velocity and pressure satisfy a discrete momentum balance equation.This correction equation is coupled with the mass and internal energy balances and the resulting system is solved. Because the internal energy balance equation is used instead of the total energy equation, corrective terms are added to ensure consistency. With an implicit-in-time first order upwinding of the numerical fluxes with respect to the material velocity only, the scheme has been proven to have a discrete solution, and to enjoy the following properties: to preserve the positivity of the density and of the internal energy, and to keep the pressure and velocity constant accross contact discontinuities [1]. In addition the scheme is asymptotic preserving is the low Mach number limit, in the sense that, with a given discretization, the computed solution is observed to converge when the Mach number tends to zero to the solution of a stable scheme for the asymptotic problem.

We build here a less diffusive variant of the scheme, by switching, for the convection fluxes in the mass and energy balances, from implicit-in-time upwind numerical fluxes to explicit-in-time A-MUSCL numerical fluxes [2, 3]. We show that under a CFL-like condition that depends only on the material velocity, the properties stated above for the former upwind scheme are preserved. In addition to that, we prove a Lax-Wendroff consistency theorem and show that the numerical solutions satisfy a discrete entropy inequality. We also observe by numerical experiments that the scheme is still able to cope with low Mach number viscous and inviscid flows. Gains in accuracy are quantified using a manufactured analytical solution.

REFERENCES

[1] Grapsas, D and Herbin, Raphaele and Kheriji, W and Latché, J.-C *An unconditionally stable staggered pressure correction scheme for the compressible Navier-Stokes equations* SMAI Journal of Computational Mathematics

[2] Piar, Libuse and Babik, Fabrice and Herbin, Raphaele and Latché, Jean-Claude *A formally second order cell centered scheme for convection-diffusion equations on unstructured non-conforming grids* International Journal for Numerical Methods in Fluids

[3] Gastaldo, L. and Herbin, R. and Latché, J.-C. and Therme, N. *A MUSCL-type segregated – explicit staggered scheme for the Euler equations* Computers and Fluids

NON LINEAR STRATEGIES FOR QUASI-INTERPOLATION.

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ABSTRACT

In this presentation, we introduce some new techniques for quasi-interpolation based on nonlinear strategies that aim to adapt to the presence of discontinuities in the data in one and several dimensions. The innovation lies in applying non-linear weights to quasi-interpolation operators of the form,

$$\mathcal{Q}(f) = \sum_{i=1}^{N} L_i(f) a_i(\mathbf{x}), \tag{1}$$

where L_i, a_i, or both form a partition of unity. We analyze different techniques where this approach can be exploited for attaining adaptation in the reconstruction of the data. The approach is of special interest when working with multivariate data and discontinuities aligned along an interface, or for designing mesh-free algorithms. The reason is the simplicity with which it is possible to extend the construction to these cases. We analyze the numerical properties of the resulting schemes, such as the smoothness, the accuracy at smooth zones and close to the discontinuity, as well as the elimination of the Gibbs phenomenon. To validate our theoretical findings, we present some numerical experiments for data in one, two, and three dimensions, which corroborate our conclusions.

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SEMI-IMPLICIT APPROACHES: FROM STIFF PROBLEMS TO DISPERSIVE EFFECTS AND SEDIMENT EVOLUTION.

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ABSTRACT

The development of efficient and robust numerical schemes for hyperbolic systems with multiscale features remains a crucial challenge in computational mathematics. Over the years, we have investigated and designed high-order semi-implicit schemes tailored for various classes of problems, including hyperbolic systems with relaxation terms, sediment transport models, and dispersive wave equations.

For relaxation systems, we have formulated semi-implicit strategies that efficiently handle stiff source terms while preserving the asymptotic properties of the system [1, 2, 3]. In sediment transport modeling, we have applied high-order semi-implicit schemes to Exner-type equations, ensuring accurate coupling between hydrodynamic and morphodynamic evolution [4, 6, 5]. More recently, our focus has shifted to dispersive systems, where we have developed and analyzed semi-implicit discretizations for hyperbolic approximations of Serre–Green–Naghdi-type equations, achieving stability and accuracy in the presence of nonlocal and nonlinear effects [7].

REFERENCES

[1] E. M. and S. Boscarino *Semi-implicit-Type Order-Adaptive CAT2 Schemes for Systems of Balance Laws with Relaxed Source Term* Communications on Applied Mathematics and Computation, 2024

[2] R. Loubere, E. M., C. Pares-Madroñal, G. Russo *CAT-MOOD Methods for Conservation Laws in One Space Dimension* SEMA SIMAI Springer Series, 2024, 35.

[3] E. M., R. Loubere, C. Pares-Madroñal, G. Russo *An almost fail-safe a-posteriori limited high-order CAT scheme* Journal of Computational Physics, 2024, 498, 112650.

[4] E. M., S. Avgerinos, M.J. Castro-Diaz, G. Russo *A semi-implicit finite volume method for the Exner model of sediment transport* Journal of Computational Physics, 2024, 499, 112714.

[5] E. M., G. Russo *Boundary effects on wave trains in the Exner model of sedimental transport* Bollettino dell'Unione Matematica Italiana, 2024, 17(2).

[6] J. Garres, E. Fernandez-Nieto, E. M., G. Russo, *High-order-Exner...* In preparation, 2025.

[7] W. Boscheri, E. M., M. Ricchiuto *Serre-Green-Naghdi-type IMEX...* In preparation, 2025.

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APPLICATIONS OF THE TR-BDF2 METHOD FOR SLOW-FAST SYSTEMS

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ABSTRACT

In this work we consider different applications of the TR-BDF2 method to slow-fast problems. Numerically, slow-fast problems need specific methods for their resolution because they correspond to so-called stiff problems. First we will present the method, as well as its main properties from the point of view of numerical analysis. It is a method with good numerical stability properties, it can be included in the methods with an associated Butcher tableaux and it is easily combinable with other methods such as finite elements in the case that the system corresponds to a partial differential equation.
We will study three versions of this method and apply it to different situations:

- Application of a combination of this method and finite element to solve a wave problem with Young's modulus with large variability.
- Adaptation of the method to a specific multirate method for the resolution of the slow-fast system relative to two point masses with a large ratio between frequencies.
- Development of the method in its stochastic version. Study of its numerical properties and its application to intracellular calcium concentrations in coupled neurons.

ACKNOWLEDGMENT

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REFERENCES

[1] L. Bonaventura, M. Gómez Mármol *The TR-BDF2 method for second order problems in structural mechanics* Computers and Mathematics with Applications 92, 13-26, Elsevier, 2021.

[2] B. Bachmann, L. Bonaventura, F. Casella, S. Fernández García, M. Gómez Mármol *Multi-rate Runge-Kutta methods: stability analysis and applications* Journal of Scientific Computing, Under review

[3] T. Caraballo Garrido, M. Gómez Mármol, I. Roldán Bocanegra *The stochastic model of Intracellular Calcium Concentrations.* In Preparation

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VARIABLE-STEPSIZE IMEX SBDF METHODS TO SOLVE ADVECTION-DIFFUSION-REACTION MODELS USING DIFFERENT SPLITTING TECHNIQUES

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ABSTRACT

Implicit-explicit (IMEX) methods are extremely popular for the solution of ordinary differential equations (ODEs) that come from the spatial discretization of partial differential equations (PDEs). A particularly challenging class of problems involves advection, diffusion, and reaction (ADR) processes because the systems are large, nonlinear and stiff. This work analyzes the application of variable stepsize, semi-implicit, backward differentiation formula (VSSBDF) methods up to fourth order to solve several ADR models when two different IMEX splitting techniques are used: physics-based splitting and dynamic linearization (Jacobian Splitting). As part of the work, the authors have developed an adaptive time-stepping and error control algorithm for VSSBDF methods up to fourth order based on a step-doubling refinement technique using estimates of the local truncation errors. The results show that a splitting approach based on dynamic linearization is an effective way of improving the performance of the VSSBDF methods in many ADR models.

INTRODUCTION

Advection-Diffusion-Reaction (ADR) problems can be mathematically modeled using partial differential equations (PDEs), which, after spatial discretization, can be represented as initial-value problems (IVPs) for systems of ordinary differential equations (ODE) in the form

$$\frac{d\mathbf{y}}{dt}(t) = \mathbf{F}(t,\mathbf{y}) = \mathbf{f}(t,\mathbf{y}) + \mathbf{g}(t,\mathbf{y}), \quad t \in [t_0, T], \quad \mathbf{y}(t_0) = \mathbf{y}_0, \tag{1}$$

where $t \in \mathbb{R}$ is time, $\mathbf{F} : \mathbb{R} \times \mathbb{R}^m \to \mathbb{R}^m$ represents the right-hand side (RHS) of the ODE which can be splitted into two parts: $\mathbf{f}$, $\mathbf{g} : \mathbb{R} \times \mathbb{R}^m \to \mathbb{R}^m$, with dimension $m \geq 1$, which are considered to be sufficiently smooth. The term $\mathbf{f}$ represents the non-stiff term, while $\mathbf{g}$ denotes the stiff term, which requires implicit integration. To take advantage of this structure, implicit–explicit (IMEX) methods that treat $\mathbf{g}$ implicitly and $\mathbf{f}$ explicitly have proven to be of interest in many studies [6]. We focus on IMEX versions of linear multistep methods (IMEX-LMM) based on backward differentiation formulae (BDF) which are highly effective for solving differential equations.

We analyze the impact of two different splitting techniques to determine functions $\mathbf{f}$ and $\mathbf{g}$ in (1):

- **Physics-based splitting**: where the terms are split based on their physical properties, e.g. $\mathbf{f}$ can represent the advection term while $\mathbf{g}$ is used for diffusion and reaction.

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- **Jacobian splitting**: where the terms are established by linearizing an IVP so that linear and non-linear terms can be integrated by different numerical techniques [1]. In this approach, $\mathbf{g}(t, \mathbf{y}(t))$ corresponds to $\mathbf{J_F}\mathbf{y}(t)$, and $\mathbf{f}(t, \mathbf{y}(t))$ corresponds to $\mathbf{F}(t, \mathbf{y}(t)) - \mathbf{J_f}\mathbf{y}(t)$, where $\mathbf{J_F}$ is the Jacobian of $\mathbf{F}$ evaluated at a specific t and $\mathbf{y}(t)$ (usually the beginning of each step t_n).

In [1] the performance of physics-based splitting and Jacobian splitting is compared on a set of ADR equations with various 2-additive Runge–Kutta methods. Building upon this work, we compare the performance of both splitting strategies on a set of six test ADR models, using 2-additive IMEX linear multistep methods up to fourth order with a variable stepsize. Recently, a comparison between IMEX-LMMs and 3-additive splitting methods (it assumes three terms instead of just two) was performed in [4].

In this work, we have employed adaptive numerical techniques to obtain the desired level of accuracy within a feasible time period and overcome the drawbacks of using a constant step size. For this purpose, we start with the family of variable stepsize semi-implicit backward differentiation formulae (VSSBDF) up to fourth order proposed in [2] and introduce an algorithm for adaptive time-stepping and error control, in which the local truncation error formula for the VSSBDF methods up to fourth order are derived as an extension of the one proposed in [3]. The proposed adaptive stepping strategy dynamically adjusts time step sizes in response to detected changes in the system's time scales. The local truncation errors for VSSBDF methods up to the fourth order are derived, utilizing a step-doubling refinement technique for approximation of these errors.

To test the effect of both splitting strategies when the VSSBDF methods are used, we have employed a suite of six ADR models, including a 1D advection-diffusion-reaction model, the CUSP model, several combustion models and the 1D and 2D Brusselator models.

CONCLUSIONS

In general, the results indicate that the VSSBDF methods utilizing Jacobian splitting outperform those utilizing the physics-based splitting in many experiments with ADR models. The results support the Jacobian splitting as a worthy splitting strategy for solving ADR IVPs.

This work is based on a previous study [5], which has been submitted to the journal Computers and Mathematics with Applications.

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REFERENCES

[1] A. Preuss, J. Lipoth, R. J. Spiteri. *When and how to split? A comparison of two IMEX splitting techniques for solving advection-diffusion-reaction equations*. Journal of Computational and Applied Mathematics, 414, 114418, 2022.

[2] Dong Wang, Steven J. Ruuth *Variable stepsize implicit-explicit linear multistep methods for time-dependent PDEs*. Journal of Computational Mathematics, 26, 838-855, 2008.

[3] David Yan, M.C. Pugh, F.P. Dawson. *Adaptive time-stepping schemes for the solution of the Poisson-Nernst-Planck equations*. Applied Numerical Mathematics. 163, 254-269, 2021.

[4] R. A. Mara'Beh, P. González Raymond J. Spiteri, J. M. Mantas. *3-additive linear multi-step methods for diffusion-reaction-advection models*. Applied Numerical Mathematics. 183, 15-38, 2023.

[5] R. A. Mara'Beh, J. M. Mantas, P. González Raymond J. Spiteri. *Performance Comparison of Variable-Stepsize IMEX SBDF Methods on Advection-Diffusion-Reaction Models*. Computers and Mathematics with Applications, 2025.

[6] S. Boscarino, L. Pareschi, G. Russo. *Implicit-Explicit Methods for Evolutionary Partial Differential Equations*. SIAM, 2025

STABILITY OF DISCRETE BOUNDARY CONDITIONS FOR LINEAR HYPERBOLIC SYSTEMS.

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ABSTRACT

This work concerns the numerical approximations of the boundary conditions for linear hyperbolic systems endowed with linear source terms. An explicit discrete boundary procedure computation is proposed to complete the three-point finite volume schemes. A proof of the fully discrete stability of the procedure is established and numerical test cases assess its relevancy.

In this work, we investigate the stability of numerical schemes for liner Initial Boundary Value hyperbolic Problems (IBVP) under the form

$$\begin{cases} \partial_t w + A\partial_x w = Qw, & \forall (x,t) \in (\mathbb{R}_*^+)^2, \\ \mathsf{N}w(0,t) = g(t), & \forall t \geq 0, \\ w(x,0) = w^{\text{in}}(x), & \forall x > 0, \end{cases} \tag{1}$$

where A is a constant $\mathbb{R}$-diagonalisable matrix, N, Q are given matrices and $g(t)$, $w^{\text{in}}(x)$ are given functions. According to [2, 3], it is well-known that (1) is well-posed if A and N satisfy the Kreiss-Lopatinskii condition that also ensures the existence of a symmetric definite matrix S and $C, \text{Kr} \geq 0$ such that

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_0^{+\infty} (w^\top S w)(x,t)\,\mathrm{d}x + \text{Kr}|\pi^- \varphi(0,t)|^2 \leq C|g(t)|^2, \tag{2}$$

where $\pi^- \varphi$ denotes the Riemann invariants related to the non-positive eigenvalues of A.

From a numerical point of view, the solutions of (1) are approximated on a grid $\left(x_{i+1/2}\right)_{i\geq 0}$ with an explicit three-point finite volume scheme writes

$$\frac{w_i^{n+1} - w_i^n}{\Delta t} + \frac{\mathfrak{F}(w_i^n, w_{i+1}^n) - \mathfrak{F}(w_{i-1}^n, w_i^n)}{\Delta x} = Qw_i^n, \quad \forall i \geq 1, \tag{3}$$

where w_i^n is the average of w on the cell i at time t^n and $\mathfrak{F}(\cdot,\cdot)$ denotes a consistent Lipchitz continuous numerical flux [1].

In this context, we propose a discretization of the boundary condition $w(0,t)$ and a viscous correction [4] for the numerical flux $\mathfrak{F}(\cdot,\cdot)$ in order to reach a fully discrete version of (2). The discrete boundary condition is defined from a reformulation of $\mathsf{N}w(0,t) = g(t)$ while the viscous correction is locally designed and it is related to an explicit definition of S in (2).

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Therefore, we obtain an explicit generic procedure to enforce the fully discrete stability for the three-point finite volume schemes (3). This stability result holds under explicit restrictions for the viscous correction and for the ratio $\frac{\Delta t}{\Delta x} > 0$. The accuracy and the stability performances of this procedure are assessed for several IBVP (1) and for several choices of $\mathfrak{F}(\cdot,\cdot)$ in (3). For instance, Figure 1 shows the results for an hyperbolic relaxation of a dispersive wave model simulated with the Lax-Wendroff numerical flux.

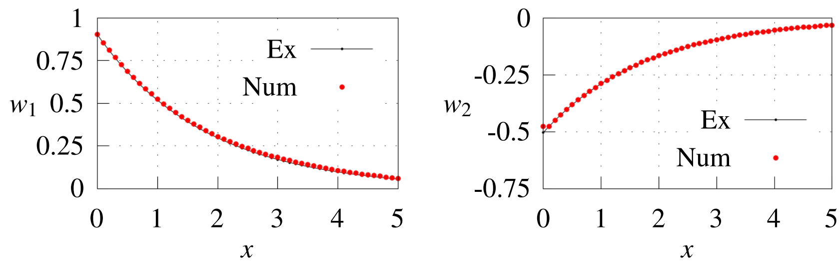

FIGURE 1: Numerical results at $t = 0.1$ for an hyperbolic relaxation of a dispersive wave model simulated with the Lax-Wendroff numerical flux in (3).

REFERENCES

[1] R. LeVeque *Numerical methods for conservation laws (2. ed.)* Birkhäuser, Lectures in mathematics, 1992.

[2] H.O. Kreiss *Initial boundary value problems for hyperbolic systems.* Communications on Pure and Applied Mathematics, vol 23, 1970.

[3] O. Guès, G. Métivier, M. Williams and K. Zumbrun *Uniform Stability Estimates for Constant-Coefficient Symmetric Hyperbolic Boundary Value Problems.* Communications in Partial Differential Equations, vol 32, Taylor & Francis, 2007.

[4] E. Tadmor *Entropy stability theory for difference approximations of nonlinear conservation laws and related time-dependent problems.* Acta Numerica, vol 12, 2003.

IMPLICIT SCHWARZ DOMAIN DECOMPOSITION METHOD WITH LEGENDRE COLLOCATION FOR A RAYLEIGH-BÉNARD PROBLEM

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ABSTRACT

INTRODUCTION

Schwarz domain decomposition methods are widely used in numerical resolution of PDE systems [2, 3]. The original domain is split into several smaller subdomains that are solved separately. In this particular work, each PDE system is solved using a Legendre collocation method [1]. Among domain decomposition, alternating methods are the most widely used [3]. There, subdomains are solved in sequence using previous values. This generates an iterative algorithm that includes some inherent error. Obviously, this error diminishes with each iteration. However, large problems with multiple subdomains can cause the stagnation of the errors [2]. To overcome this inconvenient, an implicit Schwarz domain decomposition method is introduced here. Now, every subdomain is perfectly calculated in just one iteration. Moreover, a specific algorithm can be designed to solve the corresponding implicit system in a computational time equivalent to any alternating model. In conclusion, the implicit Schwarz domain decomposition method presented achieves error-free solutions in the same amount of time as its alternating counterpart. The implicit Schwarz domain decomposition method is applied alongside a Legendre collocation method to solve the Rayleigh-Bénard convection problem.

MATHEMATICAL MODEL

The physical phenomenon studied is the Rayleigh-Bénard convection in a rectangular domain Ω This is mathematically modeled by the continuity equation, the Navier-Stokes system and the heat equation. Additionally, the Oberbeck-Boussinesq approximation is applied on the density. Finally, due to our first studies on magma and molten rock, Prandlt number is considered infinity. Then, the dimensionless equations governing the Rayleigh-Bénard convection are,

$$-\Delta \mathbf{u} + \nabla p - Ra\theta \mathbf{e}_z = 0, \qquad \nabla \cdot \mathbf{u} = 0, \qquad \partial_t \theta + \mathbf{u} \cdot \nabla \theta - \Delta \theta = 0, \tag{1}$$

where $\mathbf{u}$ is the velocity field, p the pressure, θ the temperature and Ra the Rayleigh number.

For the boundary conditions, the bottom plate is considered rigid, whereas the other three walls are non-deformable with free-slip. Furthermore, lateral walls are thermally insulated, while temperature is fixed at the bottom and upper plates.

Numerical resolution

This system is numerically solved using a second order algorithm in time and a Legendre collocation method in space [1]. Legendre collocation is a spectral method. Therefore, it is very potent and gives a global solution. However, it also is ill-conditioned, which implies that the number of collocation points cannot be increased indefinitely. In order to overcome this issue, a Schwarz domain decomposition method is applied alongside it [2].

SCHWARZ DOMAIN DECOMPOSITION

By using a Schwarz domain decomposition (SDD) method, domain Ω is split into several smaller subdomains sharing overlapping areas. Then, new PDE systems are created in each of them taking equations (1) and their corresponding boundary conditions. Additionally, Dirichlet conditions are added in the interfaces between subdomains. The remaining task is to solve these new problems using Legendre collocation. The number of collocation points in each subdomain can be controlled to avoid the ill-conditioning. However, more collocation points can be added to the whole domain by increasing the number of subdomains.

Alternating SDD

In alternating SDD, each subdomain is solved independently using the information previously known from adjacent subdomains [3]. Nevertheless, as every subdomain changes, the result will include some errors. To improve the solution, this process is repeated, creating an iterative algorithm.

Implicit SDD

We propose an implicit SDD method that solves every subdomain at the same time with a perfect transmission of the information. Therefore, the algorithm finds the exact global solution using only one iteration. On the one hand, this method does not only diminishes the errors of alternating SDD, but completely eliminates them. Moreover, due to the particular structure of the linear system generated in the implicit SDD algorithm, a specific Gauss method can be applied to decrease the computational cost significantly. On the other hand, implicit SDD loses some of the parallelizable properties that alternating SDD has.

Implicit SDD has been used to find a large variety of solutions for the Rayleigh-Bénard problem with great success, including some studies on turbulent regimes. Figure 1 includes the temperature isotherms of a particular case.

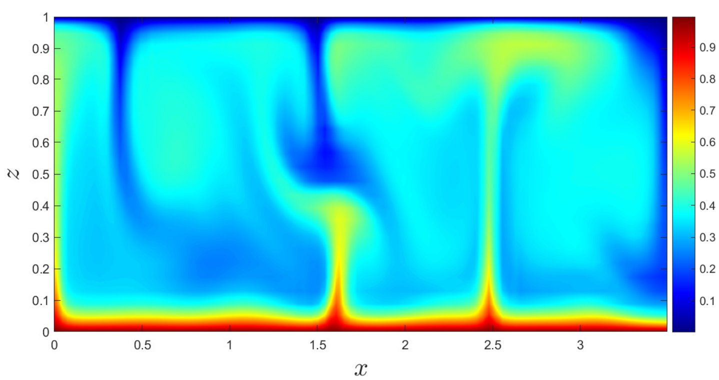

FIGURE 1: Isotherms of the Rayleigh-Bénard problem for $Ra = 4.5 \cdot 10^5$.

ACKNOWLEDGMENT

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REFERENCES

[1] C. Canuto, M. Hussaini, A. Quarteroni and T. Zang. *Spectral methods.* Springer Berlin, Heidelberg, 2007.

[2] D. Martínez, F. Pla, H. Herrero, and A. Fernández-Pérez. *A Schwarz alternating method for an evolution convection problem.* Applied Numerical mathematics 192, 179-196, 2023.

[3] H. A. Schwarz. *Uber einen Grenzübergang durch alternierendes Verfahren.* Vierteljahrsschrift der Naturforschenden Gesellschaft in Zürich 15, 272-286, 1870.

LIQUID METAL FREE SURFACE FLOW FOR MAGNETIC FUSION BLANKET.

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ABSTRACT

The standard resistive, constant density, and incompressible MHD equations define the parent model for the liquid metal under consideration. Using proper normalization, one can construct several nondimensional numbers expressing the relative strength of the different terms of the model. In particular, we explicitly introduce the Reynolds (R_e), the Hartmann(H_a), and the Magnetic Reynolds(R_m) numbers.
We are interested in situations where the dominant part of the magnetic field **B** is composed of a steady magnetic field imposed by exterior currents. This magnetic field induces currents in a moving conducting fluid, creating an induced magnetic field. The magnetic Reynolds number R_m estimates the ratio between the induced and the externally imposed magnetic field magnitudes. In Magnetic Fusion, R_m is usually very small. Therefore, the model turns to the inductionless approximation.
We will present the formal derivation of the low R_m limit model and propose a shallow approximation for thin liquid films.

INTRODUCTION

Understanding of the physics and control of thermonuclear fusion reactions has progressed in recent decades, with several fusion reactors being constructed and operated experimentally world- wide. Most explored configurations use a confinement system fueled by a Deuterium-Tricium (DT) plasma mixture. Magnetic confinement is the most advanced strategy for harnessing fusion energy for electrical power production. In this context, the DT plasma is confined by a strong magnetic field provided by superconducting magnet coils. Plasma activity is subject to instabilities [1] (i.e., edge-localize modes and disruptions) that release significant flows of electrons, neutrons, alpha par- ticles, and heat (thermal and radiative) outwards from the plasma confinement. A nuclear blanket protects the superconducting coils from the adverse effects of plasma activity and interfacing with several other components essential to the machinés operation. Liquid metal blanket face-to-plasma components offer an alternative to the most demanding protection challenges [2]. They could withstand heat fluxes without permanent damage and open the door to entirely new magnetic fusion operat ing regimes. To realize this potential, innovative technologies must be developed. Liquid lithium surfaces are an innovation that could fulfill the promise of fusion power in electricity generation[3].

MODELS AND APPROXIMATIONS.

The incompressible MHD equations describe the evolution of the velocity ($\mathbf{v}$), the pressure (p) the magnetic field $\mathbf{B}$. After normalization the system write as

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$$
\begin{aligned}
\boldsymbol{\partial}\cdot\mathbf{V} &= 0\\
\frac{\partial \mathbf{V}}{\partial t}+\boldsymbol{\partial}\cdot(\mathbf{V}\otimes\mathbf{V})+\boldsymbol{\partial} p &= \frac{H_a^2}{R_e}\mathbf{J}\times\mathbf{B}+\frac{\boldsymbol{\partial}^2\mathbf{V}}{R_e}+\boldsymbol{g}\\
\frac{\partial \mathbf{B}}{\partial t}+\boldsymbol{\partial}\times\mathbf{E} &= 0
\end{aligned}
$$

where

$$
\begin{aligned}
\nabla\cdot\mathbf{B} &= 0\\
\mathbf{J} &= \frac{\boldsymbol{\partial}\times\mathbf{B}}{R_m}=\sigma(\mathbf{E}+\mathbf{V}\times\mathbf{B})
\end{aligned}
$$

Inductionless limit.

In the limit of small magnetic Reynolds number and going back to dimensional variables, the approximation of the MHD system that we will consider is

$$
\begin{aligned}
\boldsymbol{\partial}\cdot\mathbf{V} &= 0\\
\rho\frac{\partial \mathbf{V}}{\partial t}+\rho\boldsymbol{\partial}\cdot(\mathbf{V}\otimes\mathbf{V})+\boldsymbol{\partial} p &= \mathbf{J}\times\mathbf{B}_\star+\mu\boldsymbol{\partial}^2\mathbf{V}+\boldsymbol{g}\\
\mathbf{J} &= \sigma(\boldsymbol{\partial}\varphi+\mathbf{V}\times\mathbf{B}_\star)\\
\boldsymbol{\partial}\cdot\mathbf{J} &= 0
\end{aligned}
$$

In this system $\mathbf{B}_\star$ is an imposed vector field that have to satisfy $\nabla\times\mathbf{B}_\star=0$. The current $\mathbf{J}$ is computed by Ohm's law and the constraint $\boldsymbol{\partial}\cdot\mathbf{J}=0$.

Thin flow limit in z direction.

Depth-averaged equations,

$$
\begin{aligned}
\tfrac{\partial h}{\partial t}+\nabla\cdot(h\mathbf{U}) &= 0\\
\tfrac{\partial}{\partial t}(h\mathbf{U})+\nabla\cdot(h\mathbf{U}\otimes\mathbf{U})+\nabla\left(\tfrac{\mathrm{F}_z h^2}{2}\right) &= h\mathbf{F}(\overline{\varphi},\mathbf{U})-\tfrac{h}{\rho}\nabla p_\eta\\
\nabla\cdot(h\sigma\nabla\overline{\varphi})+\nabla^\perp\cdot(\sigma\mathrm{B}_{\star,z}\mathbf{U}) &= 0\\
\mathrm{F}_z &= g_z+(\mathbf{J}_\star\times\mathbf{B}_\star)\cdot\boldsymbol{e}_z
\end{aligned}
$$

$$
\mathbf{F}(\overline{\varphi},\mathbf{U})=\overline{\boldsymbol{g}}-\mathrm{F}_z\nabla b-\sigma\frac{\mathrm{B}_{\star,z}^2\mathbf{U}-\nabla^\perp\overline{\varphi}}{\rho}+\cdots \quad\text{and}\quad p_\eta\equiv p_\star-\gamma\nabla\cdot\left(\frac{\nabla h+\nabla b}{\sqrt{1+\|\nabla h+\nabla b\|^2}}\right)
$$

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REFERENCES

[1] M Hoelzl, GTA Huijsmans, SJP Pamela, M Becoulet, E Nardon, FJ Artola, B Nkonga, et al. The JOREK non-linear extended MHD code and applications to large-scale instabilities and their control in magnetically confined fusion plasmas. *Nuclear Fusion*, 2021.

[2] M.A. Abdou et al. On the exploration of innovative concepts for fusion chamber technology. *Fusion Engineering and Design*, 2001.

[3] V. Prosta, S. Ogier-Collin, F. A. Volpea. Compact fusion blanket using plasma facing liquid Li-LiH walls and Pb pebbles. *Nuclear Materials*, 2024.

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ADAPTIVE VISCOSITY: A NUMERICAL FRAMEWORK TO IMPROVE THE ACCURACY IN CFD COMPUTATIONS.

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ABSTRACT

In this work, we present the automatic dissipation adjustment (ADA) method, and its application to several numerical schemes. These schemes are designed to enhance dissipation in under-resolved flow regions and reduce it where excessive. The resultant schemes are self-adaptive, demonstrating heightened accuracy when compared to the original scheme. Some numerical tests are performed to highlight the accuracy and robustness of the proposed numerical schemes.

INTRODUCTION

In numerical simulations of fluid flows, a certain level of dissipation is often necessary to prevent the computation from failing. However, introducing too much dissipation can compromise the accuracy of the results. Additionally, dissipation should be localized, as different regions of the computational domain may require varying amounts of dissipation at any given time. Over the past few decades, the CFD community has invested significant effort into developing numerical methods that are both very low-dissipation and robust. This has led to the creation of high-order methods, which incorporate various shock-capturing techniques to ensure stability and reliability [1, 2].

However, there are situations in the absence of shock waves where low-dissipation methods need additional viscosity to maintain accuracy. This is particularly common in under-resolved simulations of turbulent flows, or more broadly, in cases where the numerical method or computational grid fails to fully capture the flow structures [3]. In such scenarios, the numerical method may not accurately reproduce the physical process of energy transfer between scales, and without some form of dissipation, the simulation can fail. This is precisely why turbulent modeling is required in simulations of turbulent flows. Importantly, this issue is not limited to viscous turbulent flows, since it also applies to inviscid flows.

Given this, there is a clear need for numerical methods that can dynamically adjust the local amount of dissipation to maintain both accuracy and stability. From a numerical perspective, when a solution is under-resolved in a particular region of the domain, the high-frequency content of the solution variables increases[4]. This provides an opportunity to identify areas where additional dissipation is needed and to develop mechanisms for adjusting dissipation levels at specific points in the computational domain.

In this work, we introduce a numerical framework for developing self-adaptive dissipation methods based on a measure of the high-frequency content of the solution variables. We show the application of this approach to various numerical schemes, covering both inviscid and viscous flows. Our results show that this methodology has the potential to enhance the accuracy of numerical methods and opens new possibilities for their use in implicit large eddy simulation (LES) computations.

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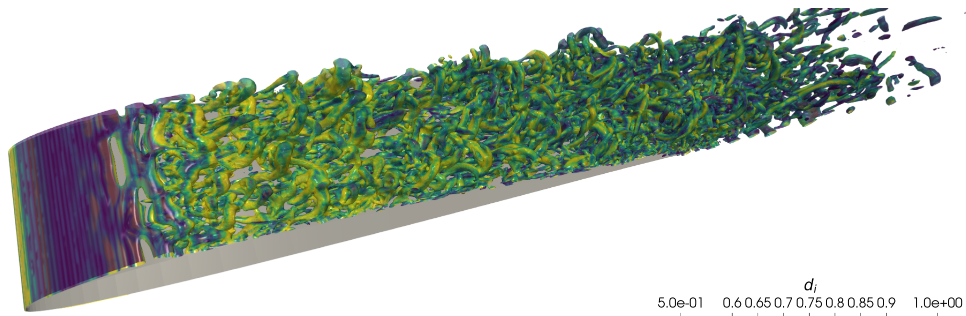

FIGURE 1: Instantaneous iso-surfaces of Q criterion (Q=100) for the SD7003 airfoil with the CWENOZ4-`ADDA` scheme. The colours indicates the different dissipation of each region. It can be seen that the `ADDA` algorithm switches to a low-dissipation d_i value in the laminar separation bubble region, while it alternates between low-dissipation and high-dissipation profile in the turbulent region according to the energy ratio values. Figure taken from [3]

ACKNOWLEDGMENTS

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REFERENCES

[1] S. Clain, S. Diot, R. Loubère *A high-order finite volume method for hyperbolic systems: Multi-dimensional Optimal Order Detection (MOOD* Journal of Computational Physics, Vol. 230(10):4028-4050, 2011.

[2] P. Tsoutsanis, M.Dumbser, *Arbitrary high order central non-oscillatory schemes on mixed-element unstructured meshes,* Computers & Fluids, Vol. 225, art. no. 104961, 2021.

[3] P. Tsoutsanis, X. Nogueira, *Arbitrary-order unstructured finite-volume methods for implicit large eddy simulation of turbulent flows with adaptive dissipation/dispersion adjustment (ADDA),* Journal of Computational Physics, 523, art. no. 113653, 2025.

[4] T. Tantikul, J.A. Domaradzki,*Large eddy simulations using truncated Navier-Stokes equations with the automatic filtering criterion,* Journal of Turbulence, 11:1-24, 2010

NUMERICAL SIMULATION OF THE 3D THERMISTOR PROBLEM IN ANISOTROPIC SEMICONDUCTOR DEVICES

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ABSTRACT

We analyze a bead type thermistor problem in an anisotropic setting and describe an approximation algorithm for its numerical simulation. The numerical results show that, in certain situations, the existence of at least three solutions.

INTRODUCTION

We consider the problem of finding two functions $u, \varphi : \bar{\Omega} \mapsto \mathbb{R}$ solution to the following nonlinear coupled system of elliptic PDEs,

$$\begin{cases} -\nabla \cdot \boldsymbol{a}(\nabla u) = \rho(u)|\nabla \varphi|^2 & \text{in } \Omega, \\ \nabla \cdot (\rho(u)\nabla \varphi) = 0 & \text{in } \Omega, \\ \boldsymbol{a}(\nabla u) \cdot \boldsymbol{n} = 0 & \text{on } \Gamma_0^- \cup \Gamma_0^+, \\ \boldsymbol{a}(\nabla u) \cdot \boldsymbol{n} = \alpha(u_{\text{ext}} - u) & \text{on } \Gamma_1, \\ \varphi = V_0 & \text{on } \Gamma_0^-, \\ \varphi = 0 & \text{on } \Gamma_0^+, \\ \dfrac{\partial \varphi}{\partial \boldsymbol{n}} = 0 & \text{on } \Gamma_1, \end{cases} \tag{1}$$

where $\Omega \subset \mathbb{R}^N$, $N \geq 1$, is an open, bounded and connected set. Its boundary $\partial\Omega$ is assumed to be smooth enough and such that $\partial\Omega = \Gamma_0^- \cup \Gamma_0^+ \cup \Gamma_1 \cup R$ where Γ_0^-, Γ_0^+ and Γ_1 are open sets in $\partial\Omega$ (with respect to the relative topology of $\partial\Omega$), smooth enough, and such that

$$\bar{\Gamma}_0^- \cap \bar{\Gamma}_0^+ = \varnothing, \quad \bar{\Gamma}_0^- \cap \Gamma_1 = \varnothing, \quad \bar{\Gamma}_0^+ \cap \Gamma_1 = \varnothing, \tag{2}$$

and $R = \partial\Omega \setminus (\Gamma_0^- \cup \Gamma_0^+ \cup \Gamma_1)$ is a set of zero hypersurface measure. The function $\boldsymbol{a} \in \mathscr{C}(\mathbb{R}^N;\mathbb{R}^N)$ is a monotone operator of algebraic vector growth degree $\vec{\boldsymbol{p}} = (p_1, \ldots, p_N)$ with $p_j \in (1, +\infty)$ for all $j \in \{1, \ldots, N\}$. The model vector field $\boldsymbol{a} = (a_1, \ldots, a_N)$ verifying (H.1)-(H.3) is given by $a_j(\zeta) = |\zeta_j|^{p_j-2}\zeta_j$ for $1 \leq j \leq N$. This leads to the $\vec{\boldsymbol{p}}$-Laplacian operator and the corresponding functional space is the anisotropic Sobolev space

$$W^{1,\vec{\boldsymbol{p}}}(\Omega) = \left\{ v \in L^{p_0}(\Omega) \,/\, \frac{\partial u}{\partial x_j} \in L^{p_j}(\Omega), \text{ for } 1 \leq j \leq N \right\}, \quad p_0 = \min\{p_1, \ldots, p_N\}. \tag{3}$$

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Also, $\rho \in \mathscr{C}(\mathbb{R})$ is such that, for some constant $\rho_1 > 0$ it holds

$$0 < \rho(s) \leq \rho_1 \text{ for all } s \in \mathbb{R}. \tag{4}$$

The mathematical analysis and numerical of the problem (1) can be found in [1, 2].

NUMERICAL RESULTS

We have performed some 3D numerical simulations for the approximation of the solutions of problem (1) in a bead type thermistor. The numerical results show that multiplicity of solutions may occur depending of the choice of the exponents p_j. For instance, by fixing to exponents and letting the third exponent close enough to 1, we obtain the existence of at least three different solutions. On the other hand, if the exponents are high enough, then we always obtain a symmetric solution.

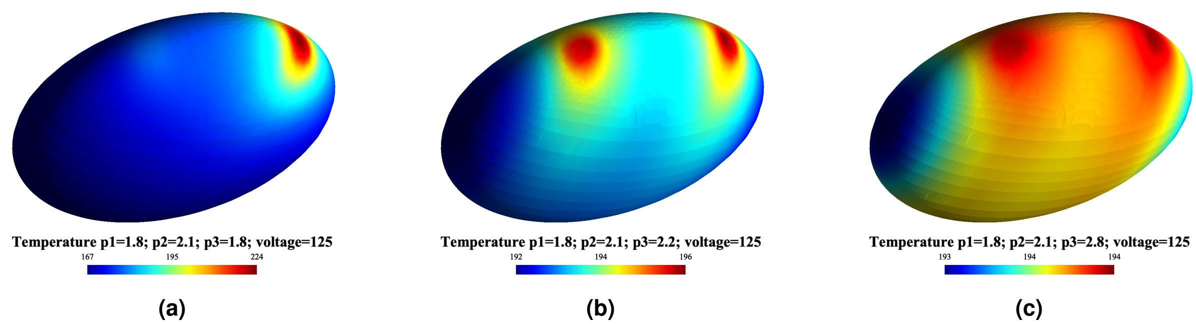

FIGURE 1: Distribution of the temperature on the thermistor surface for $u_0 = 30\,^\circ\mathrm{C}$, $V_0 = 125$ and the indicated values of the exponents p_1, p_2 and p_3.

ACKNOWLEDGMENT

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REFERENCES

[1] F. Ortegón Gallego, M. Rhoudaf, H. Talbi, *Capacity solution and numerical approximation to a nonlinear coupled elliptic system in anisotropic Sobolev spaces*, Journal of Applied Analysis and Computation, 12:6 (2022) 2184.2207. `https://doi.org/10.11948/20210208`.

[2] M. Lahrache, F. Ortegón Gallego, M. Rhoudaf *3D numerical simulation of an anisotropic bead type thermistor and multiplicity of solutions*, Mathematics and Computers in Simulation, 220 (2024) 640-672. `https://doi.org/10.1016/j.matcom.2024.02.018`.

ON THE PROJECTED HYPERBOLIC MODELS

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ABSTRACT

This talk invites the audience to consider dispersive models as hyperbolic models whose solutions are sought within a linear subspace. First we motivate this framework by describing a general strategy to derived approximate models of the water waves problem. Then we delve deeper into this mathematical structure and highlight some mathematical properties. Finally, by preserving this structure at the discrete level, we develop robust and efficient numerical schemes.

THE PROJECTED HYPERBOLIC MODELS

We focus on hyperbolic models with source term

$$\partial_t U + A(U)\,\partial_x U = -\Psi \tag{1}$$

where $U \in \mathbb{R}^{d_U}$, $\Psi \in \mathbb{R}^{d_U}$ and $A \in M_{d_U}(\mathbb{R})$ with real eigenvalues. The source term Ψ is not explicitly described, but acts to ensure that the solution remains in a linear subspace $\ker \mathscr{L}$ for a given application $\mathscr{L} : \left(L^2(\mathbb{R})\right)^{d_U} \mapsto \left(L^2(\mathbb{R})\right)^{d_Q}$. More specifically, the source term Ψ is on the dual space $\ker \mathscr{R} = (\ker \mathscr{L})^{\perp}$ such that a Helmholtz decomposition occurs.

Link with the dispersive approximations of the water waves model

Most of the approximate models of the water waves problem can be written under the projected hyperbolic form (1), such as the Korteweg–de Vries, Benjamin-Bona-Mahony and Camassa-Holm models ; the Green-Naghdi and other Boussinesq-type models, and the more complexe dispersive models with several velocities [1, 2]. More precisely, the approximate models can be recovered from a variational formulation of the water waves problem applying a Discontinuous Galerkin vertical discretization of the horizontal velocity which naturally leads to a projected hyperbolic model.

Advantages of the structure for numerical schemes

In a second time, some exemple taken advantage of the projection structure to design robust and efficient numerical schemes will be given. More precisely, by using a splitting between the hyperbolic part and the dispersive source term, the first step reads

$$U^{n*} = U^n - \delta_t A(U^n)\,\partial_x U^n \tag{2}$$

that can be solved using classical hyperbolic solver. The second step is nothing more than the Helmholtz decomposition, i.e.

$$U^{n+1} = U^{n*} - \delta_t \Psi^{n+1} \quad \text{with} \quad \mathscr{L}\left(U^{n+1}\right) = 0 \quad \text{and} \quad \mathscr{R}\left(U^{n+1}\right) = 0 \tag{3}$$

*Correspondence to martin.parisot@inria.fr which highlight the importance to preserve the duality of the operators $\mathscr{L}$ and $\mathscr{R}$ at the discrete level to ensure this step well-posed, and the dissipation of the L^2-norm is ensured. It is worth noting that the Helmholtz decomposition (3) is the most expensive part of the scheme. To reduce this cost, we propose two strategies.

The first is to use high-order scheme. The drawback of Runge-Kutta time schemes is the call of the Helmholtz decomposition (3) at each sub-time steps. Fortunately, following the case of the incompressible flow [3], the dispersive source term can be treated as a time dependent source term, reconstruct from the previous approximations, the Helmholtz decomposition being performed only at the last iteration. This strategy has beed successfully employed for the Green-Naghdi model in [4].

Most of the time and in most areas, the first hyperbolic step (2) gives a good approximation of the result. The second strategy is based on the resolution of the source term Ψ only where and when it is needed. To do so, we first propose a coupling strategy preserving the projection structure [5]. This strategy can also be used to treat discontinuous source terms, like discontinuous bathymetry, and impose practical boundary conditions. In a second time, we propose a a priori criterium selecting the model to minimize the numerical cost.

REFERENCES

[1] E. D. Fernández-Nieto, M. Parisot, Y. Penel, J. Sainte-Marie, A hierarchy of dispersive layer-averaged approximations of Euler equations for free surface flows, Communications in Mathematical Sciences 16 (5) (2018) 1169–1202. doi:10.4310/cms.2018.v16.n5.a1.

[2] C. Escalante, E. D. Fernández-Nieto, J. Garres-Díaz, T. Morales de Luna, Y. Penel, Non-hydrostatic layer-averaged approximation of euler system with enhanced dispersion properties, Computational and Applied Mathematics 42 (4) (2023) 177. doi:10.1007/s40314-023-02309-7.

[3] J. Guermond, P. Minev, J. Shen, An overview of projection methods for incompressible flows, Computer Methods in Applied Mechanics and Engineering 195 (44) (2006) 6011 – 6045. doi:https://doi.org/10.1016/j.cma.2005.10.010.

[4] M. Parisot, Entropy-satisfying scheme for a hierarchy of dispersive reduced models of free surface flow, International Journal for Numerical Methods in Fluids 91 (10) (2019) 509–531. arXiv:https://onlinelibrary.wiley.com/doi/pdf/10.1002/fld.4766, doi:10.1002/fld.4766.

[5] M. Parisot, Thick interface coupling technique for weakly dispersive models of waves, ESAIM: M2AN 58 (4) (2024) 1497–1522. doi:10.1051/m2an/2024048.

NUMERICAL METHODS FOR TWO-DIMENSIONAL POLYDISPERSE SEDIMENTATION MODELS

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ABSTRACT

A polydisperse suspension is a mixture composed of small solid particles, belonging to a number of different species that vary in size or density, which are dispersed in a viscous fluid. The modeling of the sedimentation of polydisperse suspensions, i.e., the time evolution of the concentration of each solid species by its corresponding continuity equation, is carried out by postulating that the slip velocities of each solid species (relative velocity of the solid species with respect to the liquid phase) depend on the concentrations and that the bulk velocity satisfies some incompressible fluid dynamics equations (of Navier-Stokes type) with viscosity and body forces given by the solid concentrations.

Several models for the slip velocities can be found in the literature. In this work we consider one of the most commonly used velocity models for polydisperse sedimentation is the Masliyah-Lockett-Bassoon model.

In a one-dimensional model, for which all quantities depend only on the vertical coordinate and all velocities have only vertical components, the incompressibility condition for the bulk velocity immediately implies that it is known from boundary conditions. This results in a system of first order partial differential equations whose hyperbolicity can be established in some cases.

In the multidimensional setting, besides the equations stemming from the continuity equations for each solid species, there is an extra equation for the bulk velocity. In this work, this equation is taken as a two-dimensional Stokes equation with viscosity and body forces depending on the solid concentrations.

We present a numerical scheme which is based on a finite-differences solver for the Stokes equation and perform several tests with inclined vessels to simulate the Boycott effect.

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ON THE CONSERVATION OF CURL OR DIVERGENCE CONSTRAINTS BY THE DISCONTINUOUS GALERKIN METHOD

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ABSTRACT

In this talk we are interested in *involutions* [2] in hyperbolic systems. For example, when dealing with the acoustic system

$$\begin{cases} \partial_t p + \dfrac{1}{\rho_0} \nabla_{\mathbf{x}} \cdot \mathbf{u} = 0 \\ \partial_t \mathbf{u} + \kappa_0 \nabla_{\mathbf{x}} p = 0, \end{cases}$$

where p and $\mathbf{u}$ are the pressure and the velocity, and κ_0 and ρ_0 are parameters, applying the curl to the velocity equation gives formally

$$\partial_t (\nabla \times \mathbf{u}) = 0.$$

In the same manner, considering the two-dimensional Maxwell system,

$$\begin{cases} \partial_t b + \nabla_{\mathbf{x}}^{\perp} \cdot \mathbf{e} = 0 \\ \partial_t \mathbf{e} + \nabla_{\mathbf{x}}^{\perp} b = 0, \end{cases}$$

where $\mathbf{e}$ is the electric field and b is the normal magnetic field, when the divergence is applied to the equation of electric field, we find formally

$$\partial_t (\nabla_{\mathbf{x}} \cdot \mathbf{e}) = 0.$$

Usually, a numerical scheme developed for the original conservative system does not preserve the involutions at the discrete level.

In this talk, we will begin by explaining how to preserve a curl for the acoustic system with the finite volume scheme on triangular meshes, with Godunov' solver. This case relies on a discrete Hodge-Helmholtz decomposition that can be proven to be preserved. This decomposition was introduced in [1] for Reissner-Middlin plates problems, and then was used for analyzing the accuracy problem at low Mach number [3, 4].

This example will then be rewritten in the context of a *discrete distributional* de-Rham complex [6]. Once written in this de-Rham form, we will prove that

- The classical discontinuous Galerkin method applied to the acoustic system preserves the curl on triangular meshes with Godunov' solver.
- A fully discontinuous approximation space for vectors on quadrangular meshes can be developed such that the discontinuous Galerkin method applied to the acoustic system preserves the curl on quadrangular meshes with Godunov' solver.
- Similar results exist for the preservation of the divergence.

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All these theoretical results will be illustrated by numerical examples in dimension 2 on the acoustic system, the Maxwell system and the induction system. Results of this talk can be found in the published papers [5, 7], and in the preprint [8].

REFERENCES

[1] Douglas Norman Arnold and Richard Steven Falk A uniformly accurate finite element method for the Reissner–Mindlin plate. *SIAM J. Numer. Anal.*, 26:1276–1290, 1989.

[2] Constantine Michael Dafermos Quasilinear hyperbolic systems with involutions. *Archive for Rational Mechanics and Analysis*, 94, 373-389, 1986.

[3] Stephane Dellacherie, Pascal Omnes, and Felix Rieper. The influence of cell geometry on the Godunov scheme applied to the linear wave equation. *Journal of Computational Physics*, 229(14):5315–5338, 2010.

[4] Jonathan Jung and Vincent Perrier. Steady low Mach number flows: identification of the spurious mode and filtering method. *Journal of Computational Physics*, 468, 111462, 2022.

[5] Jonathan Jung and Vincent Perrier. A curl preserving finite volume scheme by space velocity enrichment. Application to the low Mach number accuracy problem. *Journal of Computational Physics*, 515, 113252, 2024.

[6] Martin Werner Licht. Complexes of discrete distributional differential forms and their homology theory. *Foundations of Computational Mathematics*, 17(4):1085–1122, 2017.

[7] Vincent Perrier. discrete de-Rham complex involving a discontinuous finite element space for velocities: the case of periodic straight triangular and Cartesian meshes. *Annales Henri Lebesgue*, accepted, available at https://inria.hal.science/hal-04564069v3/.

[8] Vincent Perrier Development of discontinuous Galerkin methods for hyperbolic systems that preserve a curl or a divergence constraint: the case of linear systems, preprint, available at https://inria.hal.science/hal-04564886v2, 2025.

IN-CELL DISCONTINUOUS RECONSTRUCTION PATH-CONSERVATIVE METHODS FOR CONSERVATIVE SYSTEMS IN NONCONSERVATIVE FORM

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ABSTRACT

We consider nonconservative hyperbolic systems whose conservative form is available. We are interested in the numerical approximation of discontinuous solutions for these systems. Due to the lack of control of numerical viscosity, standard path-conservative methods converge, in general, to weak solutions with wrong jump conditions in the presence of non-conservative products. The main idea of this work is to use the strategy developed in [1, 2, 3] and combine it with the good jump conditions given by the conservative form of the systems. The advantage is that the governing equations can be solved directly in the most physically relevant set of variables, the primitive variables of the nonconservative system. We will show some numerical results for the Euler multi-material system comparing path-conservative methods with and without this in-cell discontinuous reconstruction approach and the fully conservative methods. First- and second-order methods will be considered.

ACKNOWLEDGMENT

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REFERENCES

[1] C. Chalons *Path-conservative in-cell discontinuous reconstruction schemes for non conservative hyperbolic system* Commun. Math. Sci., vol 18, International Press, 2020.

[2] E. Pimentel-García, M. J. Castro, C. Chalons, T. Morales de Luna, and C. Parés *In-cell discontinuous reconstruc- tion path-conservative methods for non conservative hyperbolic systems - Second-order extension* Journal of Computational Physics, vol 459, Elsevier, 2022.

[3] E. Pimentel-García, M. J. Castro, C. Chalons, and C. Parés *High-order in-cell discontinuous reconstruction path-conservative methods for nonconservative hyperbolic systems–DR.MOOD method* Numerical Methods for Partial Differential Equations, vol 40, John Wiley & Sons, Inc., 2024.

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ON SOME GENERAL MULTIDIMENSIONAL, MULTI-STATE RIEMANN SOLVERS FOR HYPERBOLIC SYSTEMS OF CONSERVATION LAWS

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ABSTRACT

We introduce a framework to design multidimensional Riemann solvers for nonlinear systems of hyperbolic conservation laws on unstructured Voronoi-like tessellations. In this framework we propose two simple but complete solvers. The first method is a direct extension of the Osher-Solomon Riemann solver to multiple space dimensions. The second method is a genuinely multidimensional upwind flux, with properties similar to the residual distribution N-scheme, P.L. Roe's original optimal multi-dimensional upwind advection scheme. Both methods use the full eigenstructure of the underlying hyperbolic system and are therefore complete by construction. A simple higher order extension up to fourth order in space and time is then introduced at the aid of a CWENO reconstruction in space and an ADER approach in time, leading to a family of high order accurate fully-discrete one-step schemes based on genuinely multidimensional Riemann solvers. We present applications of our new numerical schemes to several test problems for the compressible Euler equations of gas-dyanamics. In particular, we show that the proposed schemes are at the same time carbuncle-free and preserve certain stationary shear waves exactly.

INTRODUCTION

Our work follows the quest of designing genuinely multidimensional Riemann solvers which may generalize the work of Godunov [?] in 1959. This quest started with the initial attempts by Roe *et al.* [?, ?], Colella [?], Saltzman [?] and many others. Many contributed to this question. We mention the contributions by Balsara *et al.* leaing to multidimensional approximate Riemann solvers of the HLL-type, see [?, ?] and references therein. Very recently [?] to the Eulerian description of compressible gas dynamics on fixed grids multidimensional nodal solvers commonly used in Lagrange hydrodynamics. Other works attempted to go beyond the Riemann Problem setting. This is the case of the contributions due to joint efforts of groups revolving around P.L. Roe and H. Deconinck which have led to the schemes today known as residual distribution (RD) [?, ?]. Similar motivation also other families of methods, as the Active Flux schemes (see e.g. the recent lecture [?]).

PRESENT CONTRIBUTION

In this work we combine ideas from the multidimensional upwind RD and multidimensional Riemann solver frameworks. We work in the setting of unstructured meshes composed of nodal Voronoi-like cells Ω_c obtained by joining the gravity centers of the triangles around each node p (cf. Fig.(a) below). The design or our schemes exploit explicitly this geometrical definition. In particular, we work with a prototype which can be written as

$$\mathbf{Q}_c^{n+1} = \mathbf{Q}_c^n - \frac{\Delta t}{|\Omega_c|} + \sum_{p \in \partial\Omega_c} \widehat{\mathbf{F}}_{pc} \cdot \mathbf{n}_{pc}$$

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where $\widehat{\mathbf{F}}_{pc} \cdot \mathbf{n}_{pc}$ is a unique numerical corner flux, along the corner normal $\mathbf{n}_{pc}$, taking as input $d+1$ states. In two dimensions, these states are associated to the cells of the triangle T_p containing point p (cf. Fig(b) below).

To design $\widehat{\mathbf{F}}_{pc} \cdot \mathbf{n}_{pc}$ we rely on the existence of the simplex T_p. On one hand this can be used to evaluate multidimensional solution gradients, which are used to design a direct extension of the Osher-Solomon Riemann solver. Here, the multidimensional numerical dissipation is obtained by integrating the absolute value of the flux Jacobians over a dual triangular mesh around each node of the primal polygonal grid. The required nodal gradient is then evaluated on a local computational simplex involving the $d+1$ neighbours meeting at each corner. On the other, this allows to establish an equivalence between the prototype above and residual distribution. This equivalence is used to design a numerical flux starting from the so-called N-scheme, P.L. Roe's original optimal multi-dimensional upwind advection scheme.

High order extensions are obtained using CWENO reconstructions to evaluate the states for the Riemann solver, and high order ADER time stepping. The resulting schemes have reduced dissipation, and enhanced approximation of contact discontinuities. This can be seen in figures Fig(c) and Fig (d) reporting WENO1 solutions for two-dimensional Riemann problem.

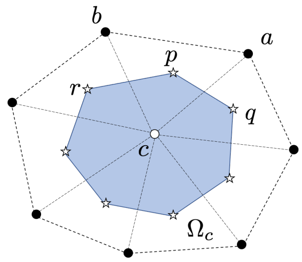

Fig. (a)

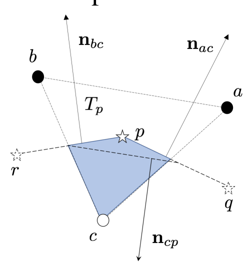

Fig. (b)

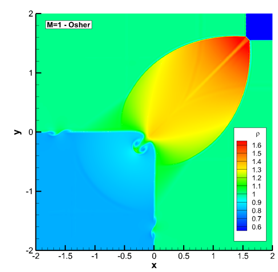

Fig. (c)

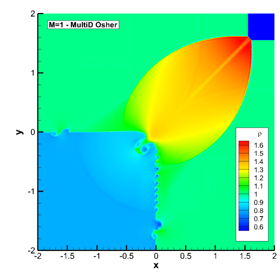

Fig. (d)

REFERENCES

[1] S.K. Godunov *Finite Difference Methods for the Computation of Discontinuous Solutions of the Equations of Fluid Dynamics* Mathematics of the USSR: Sbornik, 1959

[2] P.L. Roe *Discrete models for the numerical analysis of time–dependent multidimensional gas dynamics* J.Comput.Phys., 1986

[3] C.L. Rumsey and B. van Leer and P.L. Roe *A multidimensional flux function with applications to the Euler and Navier-Stokes equations* J.Comput.Phys., 1993

[4] P. Colella *Multidimensional upwind methods for hyperbolic conservation laws* J.Comput.Phys., 1990

[5] J. Saltzman *An unsplit 3D upwind method for hyperbolic conservation laws* J.Comput.Phys., 1994

[6] D.S. Balsara *Multidimensional HLLE Riemann solver: Application to Euler and magnetohydrodynamic flows* J.Comput.Phys., 2010

[7] D.S. Balsara and M. Dumbser *Divergence-free MHD on unstructured meshes using high order finite volume schemes based on multidimensional Riemann solvers* J.Comput.Phys., 2015

[8] G. Gallice and A. Chan and R. Loubère and P.H. Maire *Entropy stable and positivity preserving Godunov-type schemes for multidimensional hyperbolic systems on unstructured grid* J.Comput.Phys., 2022

[9] H. Deconinck and M. Ricchiuto *Residual Distribution Schemes: Foundations and Analysis* Encyclopedia of Computational Mechanics, 2007

[10] R. Abgrall and M. Ricchiuto *Hyperbolic balance laws: residual distribution, local and global fluxes* In: Numerical Fluid Dynamics, Methods and Computations, Springer 2022

[11] P.L. Roe *Musings of a Computational Phylosopher* In: Proc.s Cambridge Unsteady Flow Symposium 2024, Springer 2024

SEMI-IMPLICIT HYBRID SCHEMES FOR CONTINUUM MECHANICS ON UNSTRUCTURED GRIDS

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ABSTRACT

In this talk, we present novel numerical methods for the solution of the Godunov-Peshkov-Romenski (GPR) model for continuum mechanics [1, 2] based on a splitting operator strategy that decouples the original first order hyperbolic system into three subsystems. The approaches designed lead to a mild CFL restriction for the overall scheme with a time step independent of the sound velocity and of the potentially stiff source terms [3]. The first subsystem, solved explicitly using a finite volume approach, gathers the convective terms and the non-conservative products. Then, the contributions due to the stiff source terms on the distortion, thermal impulse and energy equations are locally computed at each cell using an implicit DIRK method. Finally, a Poisson type subsystem, which yields the total energy and the corresponding correction term for the momentum, is solved using continuous finite elements. The proposed methodology is assessed for a large set of test cases both in the solid and fluid limits of the model and considering a wide range of Mach numbers.

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REFERENCES

[1] I. Peshkov, E. Romenski. *A hyperbolic model for viscous Newtonian flows.* Continuum Mechanics and Thermodynamics, vol 28, 85-104, Springer, 2016.

[2] M. Dumbser, I. Peshkov, E. Romenski, O. Zanotti. *High order ADER schemes for a unified first order hyperbolic formulation of continuum mechanics: Viscous heat-conducting fluids and elastic solids.* Journal of Computational Physics, vol 314, 824-862, Springer, 2016.

[3] S. Busto, L. Río-Martín. *Semi-implicit hybrid finite volume/finite element method for the GPR model of continuum mechanics.* Journal of Scientific Computing, vol 102, 49, Springer, 2025.

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NUMERICAL APPROXIMATION OF NON-CONVEX SPECIAL RELATIVISTIC HYDRODYNAMICS

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ABSTRACT

The equations of special relativistic hydrodynamics (SRHD) form a nonlinear system of conservation laws which is closed with an equation of state (EoS) characterizing the equilibrium thermodynamic properties of the considered material. The thermodynamics, through the EoS, provides the classical or nonclassical (convex or non-convex) character of the wave structure. In non-convex dynamics, in addition to elementary waves such as rarefactions and shocks, combinations of these can appear developing complex structures.

In order to simulate complex dynamics in SRHD in which phase changes appear (as in high-energy astrophysical events like coalescence of binaries of black holes) it is required that thermodynamics be represented by non-convex equations of state [1].

In this research work we deal with the numerical approximation of the complex structure in SRHD when the system is closed with a non-convex equation of state. We consider a phenomenological relativistic EoS that mimics the loss of classical behavior when the fluid enters into a non-convex—thermodynamically—region in the relativistic regime [4].

We study the numerical approximation of the high order versions of several widely used shock capturing numerical schemes for relativistic hydrodynamics [3, 5, 6]. We analyze their behaviour for different prescribed viscosities comparing their approximation against the exact solution of Riemann problems developing non-convex dynamics presented in [2]. In particular we examine the accuracy of the numerical approximation of the methods in the density shell of two blast wave problems where composite waves are formed. We also evaluate the stability for highly relativistic colliding slabs problems.

REFERENCES

[1] Aloy, M.A, Ibanez, JM,Sanchis-Gual,N, Obergaulinger,M, Font, JA, Serna, S and Marquina, A, *Neutron star collapse and gravitational waves with a non-convex equation of state*, Monthly Notices of the Royal Astronomical Society, 484(4), 4980–5008, 2019.

[2] Berbel, M., Serna, S., Marquina, A., *Exact Riemann solver for nonconvex relativistic hydrodynamics*, Journal of Fluid Mechanics, 975, 2023.

[3] Harten, A., Lax, P.D., van Leer, B, *On upstream differencing and Godunov-type schemes for hyperbolic conservation laws*, SIAM review , 25(1), 35–61, 1983.

[4] Ibanez, J.M., Marquina, A., Serna, S., Aloy, M.A, *Anomalous dynamics triggered by a non-convex equation of state in relativistic flows*, Monthly Notices of the Royal Astronomical Society, 476, 1100–1110, 2017.

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[5] Marquina, A., Serna, S., Ibanez, J.M., *Capturing Composite Waves in Non-convex Special Relativistic Hydrodynamics*, Journal of Scientific Computing , 2132–2161, 2019.

[6] Toro, E.F., Spruce, M., Speares, W., *Restoration of the contact surface in the HLL-Riemann solver*, Shock waves, 4, 25–34, 1994.

A DATA-DRIVEN ROM METHOD FOR NETWORKS OF MULTISCALE DYNAMICAL SYSTEMS

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ABSTRACT

In this talk, we present the novel technique introduced in [2], based on the Proper Orthogonal Decomposition method, for dynamical systems with multiple timescales.

AUTOMATIC PROPER ORTHOGONAL BLOCK DECOMPOSITION

In the article [1], we consider Reduced Order Models in order to save computational costs in simulations of the multiple timescale network model of intracellular calcium concentrations in coupled neurons [3]. Despite the ability of these intrusive methods to approximate closely the solution generated by the original system, performing these techniques may make us lose the original structure of the problem, such as, for instance, the slow–fast separation of variables typical in neuroscience problems. To tackle this problem, we present a novel technique introduced in [2], based on the Proper Orthogonal Decomposition method, for dynamical systems with multiple timescales. The main ideas are to retain the structure of the original model, which is lost in the original POD procedure, while producing a competitive reduction in the number of equations and computational time, and to determine the best structure. We present some numerical tests for various behaviors of three different neural network models with multiple timescales, which support the use of these new methods.

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REFERENCES

[1] A. Bandera, S. Fernández-García, M. Gómez-Mármol, & A. Vidal, *A multiple timescale network model of intracellular calcium concentrations in coupled neurons: Insights from ROM simulations* Mathematical Modelling of Natural Phenomena, vol 17, 11, 2022.

[2] A. Bandera, S. Fernández-García, M. Gómez-Mármol, & A. Vidal, *Automatic Proper Orthogonal Block Decomposition method for network dynamical systems with multiple timescales* Communications in Nonlinear Science and Numerical Simulation, 107844, 2024.

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[3] S. Fernández-García & A. Vidal, *Symmetric coupling of multiple timescale systems with mixed-mode oscillations and synchronization*Physica D, 401,132129,2020

IMPLEMENTATION OF A DETERMINISTIC SOLVER FOR DG-MOSFETS ON A HIGH-PERFORMANCE PLATFORM

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ABSTRACT

The goal of this project is to develop efficient tools for the simulation of an ultra-short (10 nm) Double-Gate Metal Oxide Semiconductor Field-Effect Transistor (DG-MOSFET) through a deterministic solver operating at mesoscopic level, to provide reference results to the community of electronic engineers, who rely more on Monte-Carlo and macroscopic solvers. The work we present comes as a completion of several steps. In 2009 the paper [1] was published, in which a simplified model and a sequential code are used. That work was followed in 2014 by [2], in which scattering phenomena were added and an MPI parallelization was developed. Then in 2020 the paper [3] was published, in which the code was partially ported to Nvidia graphic cards. Finally, in 2023 in [4] the port to GPU has been completed.

REFERENCES

[1] Naoufel Ben Abdallah, María J. Cáceres, José Antonio Carrillo, Francesco Vecil, *A deterministic solver for a hybrid quantum-classical transport model in nanoMOSFETs*, Journal of Computational Physics, vol 228, 2009

[2] Francesco Vecil, José Miguel Mantas, María J. Cáceres, Carlos Sampedro, Andrés Godoy, Francisco Gámiz, *A parallel deterministic solver for the Schrödinger-Poisson-Boltzmann system in ultra-short DG-MOSFETs: Comparison with Monte Carlo*, Computers and Mathematics with Applications, vol 67, 2014

[3] José Miguel Mantas, Francesco Vecil, *Hybrid OpenMP-CUDA parallel implementation of a deterministic solver for ultrashort DG-MOSFETs*, The Internation Journal of High Performance Computing Applications, vol 34, 2020

[4] Francesco Vecil, José Miguel Mantas, Pedro Alonso-Jordá, *Efficient GPU implementation of a Boltzmann-Schrödinger-Poisson solver for the simulation of nanoscale DG MOSFETs*, The Journal of Supercomputing, 79, 2023

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List of posters

R-BLOCK STRUCTURAL SCHEMES OF HIGH ORDER ACCURACY

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ABSTRACT

We introduce the compact structural schemes that considers a set of Physical Equations and Structural Equations. The Physical Equations utilize the function and its K derivatives at a given node through the implementation of physical relations. On the other hand, the Structural Equations are based on linear relationships between the function and its derivatives over a stencil of R points, referred to as a R-block, which defines full connections between a node and its neighboring nodes. These relationships are independent of the specific physics being modeled.

After a short presentation of some structural schemes, we prove their unconditional stability and excellent behaviour in terms of spectral resolution. Numerical examples will confirm the schemes ability to provide accurate and stable solutions.

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HYBRID REDUCED ORDER MODEL FOR HEAT EXCHANGE IN CONCENTRATED SOLAR POWER RECEIVERS

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ABSTRACT

Hybrid ROM for Heat Exchange for Heat Transfer Fluid. We present a Proper Orthogonal Decomposition-Reduced Order Model (POD-ROM) approach to the Heat Transfer Fluid (HTF) in the mod- elling of Concentrated Solar Power (CSP) tower receivers. We build a 3D hybrid POD-ROM combining two techniques, an intrusive one for the temper- ature field and a non-intrusive data-driven one for the velocity field. Addition- ally, two different software are employed: Ansys Fluent, which is a commercial software, and FreeFEM, an open-source software. Accurate results are obtained with relative errors of 103 and a speedup of 4 orders of magnitude

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ENHANCING SPANISH TSUNAMI EARLY WARNINGS BY ACCURATELY PREDICTING KEY PARAMETERS

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ABSTRACT

Tsunami Early Warning Systems (TEWS) play a crucial role in minimizing the impact of tsunamis on coastal communities globally. In the NEAM region (North-East Atlantic, the Mediterranean, and connected Seas), historical approaches involve using Decision Matrices and precomputed databases due to the short time between tsunami generation and coastal impact. Overcoming real-time simulation challenges, the EDANYA group at the University of Málaga developed Tsunami-HySEA, a GPU code enabling Faster Than Real Time (FTRT) tsunami simulations. This code is successfully implemented and tested in TEWS of countries like Spain, Italy, and Chile, this code has undergone rigorous verification and validation processes.

In collaboration with the National Geographic Institute of Spain, we have extended the work previously done where we take advantage of the machine learning techniques and proposed a first approach to the use of neural networks (NN) to predict the maximum wave height and arrival time of tsunamis in the context of TEWS with very good results. This approach offers the advantage of minimal inference time and can be executed on any computer. It accommodates uncertain input data, delivering results within seconds.

As tsunamis are rare events, numerical simulations using the Tsunami-HySEA are used to train the NN model. This phase demands numerous simulations, necessitating substantial High-Performance Computing (HPC) resources. Approximately 300,000 simulations have been done to cover different faults in the Atlantic Ocean.

The goal is to develop neural network models for predicting the maximum wave height of such tsunamis at multiple coastal locations simultaneously. To cover Huelva and Cádiz coast, 78 points in the coastline have been selected for their predictions. The main importance of this work is that the models developed will be implemented in the Spanish TEWS which will produce an estimation of the tsunami impact in seconds.

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REFERENCES

[1] B. Gaite, J. Macías, J.V. Cantavella, C. Sánchez-Linares, C. González, and L.C. Puertas *Analysis of Faster-Than-Real-Time (FTRT) Tsunami Simulations for the Spanish Tsunami Warning System for the Atlantic* GeoHazards, vol. 3, no. 3, pp. 371–394, 2022.

[2] J.F. Rodríguez, J. Macías, M.J. Castro, M. de la Asunción, and C. Sánchez-Linares *Use of Neural Networks for Tsunami Maximum Height and Arrival Time Predictions* GeoHazards, vol. 3, no. 2, pp. 323–344, 2022.

NEURAL NETWORK-BASED IMPLICIT FINITE VOLUME SCHEMES FOR HYPERBOLIC SYSTEM OF CONSERVATION LAWS

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ABSTRACT

The rapid evolution of Physics Informed Neural Networks (PINNs) [1] and Variational PINNs (VPINNs) [2] signifies a transformative shift in the field of computational mathematics, particularly in solving complex non-linear partial differential equations (PDEs). These methodologies efficiently encode physical laws into the architecture of neural networks, allowing the numerical approximation of PDEs.

In this context, we propose to combine PINNs and the traditional finite volume methods (FVM), which are widely used for numerical approximations of hyperbolic conservation laws. First, we consider the integral form of the hyperbolic conservation law and a suitable partition of our computational mesh into cells. Next, we will use neural networks in each cell as a reconstruction operator in the FVM framework. The advantage of using neural networks is to be able to consider implicit methods without increasing the complexity, which allows us to increase stability and to be able to use larger time steps. The neural network performs the crucial task of reconstructing intercell fluxes. The way the method is assembled will allow us to construct well-balanced methods in this framework [3], or to make use of entropy-conservative numerical fluxes.

Moreover, the efficiency of the proposed method will be demonstrated through its application to a variety of hyperbolic conservation laws, including Burgers' equations, shallow water equations, and the compressible Euler equations.

REFERENCES

[1] S. Mishra and R. Molinaro, *Estimates on the generalization error of physics-informed neural networks for approximating PDEs*. IMA Journal of Numerical Analysis, 2022.

[2] T. Rick and S. Mishra and R. Molinaro, *wPINNs: Weak Physics informed neural networks for approximating entropy solutions of hyperbolic conservation laws*. arXiv, 2022.

[3] M.J. Castro and T. Morales de Luna and C. Parés, *Well-Balanced Schemes and Path-Conservative Numerical Methods*. Elsevier, 2017.

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A GENERAL MODEL FOR SHALLOW WATER EQUATIONS IN SPHERICAL COORDINATES

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ABSTRACT

When modelling large-scale phenomena like tsunamis, the curvature of the Earth must be taken into account. For that reason, the shallow water equations in spherical coordinates have been widely used as a tool for simulating this type of events. In this work we derive a general shallow water system in spherical coordinates. In order to obtain this model, we follow the usual approach for the shallow water equations in Cartesian coordinates, which are derived by averaging Euler equations in height. In our case, we do an averaging process in the normal direction of the sphere using Euler equations in spherical coordinates.

Once the general model is derived, we compare it with the simplified shallow water equations in spherical coordinates that have been used in works like [3], [1], [2]. Doing a re-escalation process we show how the system derived is more general that the one used in the literature.

Furthermore, the more general shallow water system can be rewritten in a conservative form so that the mass of water is conserved. This is in fact an improvement in comparison with the simplified system, which, as shown in [1], only conserves the height of the column of water.

Some numerical test are done in order to compare the results given by both systems. When the radius of the planet is not large in comparison with the length of the bathymetry and the length of the column of water we can see noticeable differences between both systems. However, almost not perceptible differences can be found when the radius of the planet is larger enough in comparison with the other two variables.

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REFERENCES

[1] Manuel J Castro, Sergio Ortega, and Carlos Parés. Well-balanced methods for the shallow water equations in spherical coordinates. *Computers & Fluids*, 157:196–207, 2017.

[2] Matthias Läuter, Dörthe Handorf, and Klaus Dethloff. Unsteady analytical solutions of the spherical shallow water equations. *Journal of computational physics*, 210(2):535–553, 2005.

[3] Wei-Yan Tan. *Shallow water hydrodynamics: Mathematical theory and numerical solution for a two-dimensional system of shallow-water equations*. Elsevier, 1992.

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BAROTROPIC-BAROCLINIC SPLITTING FOR MULTI-LAYER ROTATING SHALLOW WATER MODELS

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ABSTRACT

The classical shallow water equations have a limited range of application because they cannot model vertical effects. The multi-layer approach ([1] and others) allows to recover some vertical variations of the flow while keeping some simplification of the shallow water model. The computational costs obviously grow with the number of layers, which is often around 50 in ocean simulations. With a large computational domain and/or long time scales and a focus on the evolution on tracers rather than on the free surface, a strategy should be employed to reduce the computational cost.

In this contribution we focus on the barotropic-baroclinic splitting which is employed in (numerical) ocean models [2]. The fast barotropic gravity waves are treated in a vertically averaged manner. The slower baroclinic dynamic is fully multidimensional but has much larger time step. We reformulate this strategy in the nonlinear multi-layer framework and write it as an exact splitting which behaves well regarding the total energy. No filters or corrections are needed. The barotropic step gathers the evolution of the free surface and the mean velocity with an accurate and well-balanced one layer shallow water model. The baroclinic step includes the vertical exchange between the layers and an adjustment of the velocities around their mean value. In addition to drastically reducing the computational costs, the scheme also reduces the amount of numerical dissipation.

Currently this work deals with the constant density case, but in ongoing work we are extending the barotropic-baroclinic splitting to the variable density case in order to model situations such as coastal upwelling.

REFERENCES

[1] E. Audusse, M.-O. Bristeau, B. Perthame and J. Sainte-Marie. "A multilayer Saint-Venant system with mass exchanges for shallow water flows. Derivation and numerical validation", *ESAIM. M2AN*, 45, 2011.

[2] P. D. Killworth, D. J. Webb, D. Stainforth and S. M. Paterson. "The Development of a Free-Surface Bryan–Cox–Semtner Ocean Model.", *JPO*, 1991.

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BOUNDARY TREATMENT FOR HIGH-ORDER IMEX RUNGE–KUTTA LOCAL DISCONTINUOUS GALERKIN SCHEMES FOR MULTIDIMENSIONAL NONLINEAR PARABOLIC PDES

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ABSTRACT

In this work, we introduce new boundary treatment algorithms that prevent order reduction in implicit-explicit Runge-Kutta time discretizations for convection-diffusion-reaction problems with time-dependent Dirichlet boundary conditions. Our focus lies on Cartesian meshes and equations with stiff diffusive components. The proposed methods handle boundary values in the implicit-explicit internal stages in the same manner as interior points, and they are designed for multidimensional problems that may include nonlinear advection and source terms. Numerical experiments confirm that the methods achieve their intended order of convergence. While our spatial discretization is performed in the local discontinuous Galerkin framework, the boundary treatment algorithms can be used with other spatial schemes, including finite differences, finite elements, and finite volumes [1, 2].

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REFERENCES

[1] H. Wang, Q. Zhang, and C.-W. Shu. *Third order implicit-explicit Runge-Kutta local dis-continuous Galerkin methods with suitable boundary treatment for convection-diffusion problems with Dirichlet boundary conditions*, J. Comput. Appl. Math., 342, 2018, pp. 164-179.

[2] V. González-Tabernero, J. G. López-Salas, M. J. Castro-Díaz, and J. A. García-Rodríguez, *Boundary Treatment for High-Order IMEX Runge–Kutta Local Discontinuous Galerkin Schemes for Multidimensional Nonlinear Parabolic PDEs*, SIAM Journal on Scientific Computing, 43 (5), 2024.

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City of Málaga

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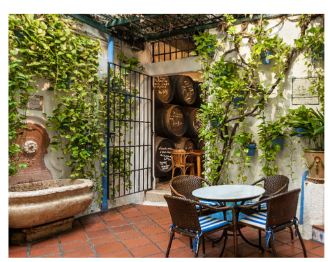

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